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COMPUTATION OF EIGENVALUES AND EIGENVECTORS OF NONCLASSICALLY DAMPED SYSTEMS

S. S. Peres-Da-Silva,† **D. L. Cronin**‡§ and **T. W. Randolph**¶

†EASi Engineering, Bingham Farms, MI 48025, U.S.A.

‡Department of Mechanical and Aerospace Engineering and Engineering Mechanics, University of Missouri-Rolla, Rolla, MO 65401, U.S.A.

¶Department of Mathematics and Statistics, University of Missouri-Rolla, Rolla, MO 65401, U.S.A.

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Abstract—Conventionally, the eigenanalysis of a nonclassically damped dynamic system is performed in a space of twice the system's dimension. This and the properties of the matrices characterizing the system in this space make the analysis costly, particularly for large systems. Prior to the development several years ago by Cronin of a new computational method, there was no alternative to the conventional analysis. The convergence of the new method was not established then by Cronin, but he illustrated it by analyzing a number of representative systems. We set out in a present work to develop a predictor of convergence for the new method, and observed that a subtle revision of the method leads to a rigorous and useful convergence condition. The revised method for eigenanalysis is derived here, as is its convergence condition. Illustrative worked examples are included, notably an example involving a gyroscopic system that illustrates the utility of the method for the case of a non-symmetric damping matrix.

1. INTRODUCTION

To determine the free or forced vibration of large dynamic systems, efficient numerical analysis first requires that an eigenanalysis be performed. This process is low in cost if the system is proportionally damped [1], or, more generally, if it is classically damped [2]. If the system is nonclassical, the eigenanalysis becomes relatively expensive because it is conventionally performed in a space of twice the system's dimension, and because, depending on the formulation used, the matrices characterizing the system in the new space are either nonsymmetric or non-positive definite.

The high relative cost of the eigenanalysis of nonclassically damped systems is possibly a motivation for investigators who explored alternate means for the exact analysis of such systems, [3, 4], and their approximate analysis, [5–7]. A series approach is common to these investigations. The approximate work led to the first one or two terms of series descriptions for the eigenvalues and eigenvectors. The exact analysis produced an expression for the general term of each series.

Although Cronin, the author of the exact analysis, illustrated the convergence of his method by examining a number of dynamic systems, he provided no means to determine whether (or how well) the series will converge for a given dynamic system. Related, of course, is the matter of estimating the quality of the various approximations described in the above references. The authors involved offered no estimators of quality.

In the present paper we examine rigorously the matter of the convergence of perturbation series solutions to the perturbed special eigenvalue problem as discussed by Kato [10], and we develop a simple convergence condition for the case of a normal unperturbed matrix. We transform the eigenvalue problem for a nonclassically damped system into the form of a special eigenvalue problem. We observe, thereby, that the exact series analysis of Refs [3, 4] requires a subtle reformulation before a simple convergence condition can be used. We do the needed reformulation and, using the steps given in the above references, we produce new expressions for the general terms of the eigenvalue series and the eigenvector series. We also state in appropriate terms the convergence condition for these series.

We include several examples to show how the new series behave and how the convergence condition works. Since our approach is intended to be suitable for nonsymmetric damping matrices, we illustrate this with the example of a gyroscopic system.

§ To whom all correspondence should be addressed.

2. ANALYSIS

2.1. *Discussion*

Given the assumption of linearity, the homogeneous equations of motion for an nth order dynamic system are:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{0\} \tag{1}$$

where $[M]$, $[C]$ and $[K]$ are real $(n \times n)$ matrices describing the mass, damping and stiffness properties of the system, respectively. The $(n \times 1)$ vector $\{x\}$ describes the system's displacements. The mass and stiffness matrices are assumed to be symmetric and positive definite. No special properties are assumed for the damping matrix.

The eigenanalysis of the system described by eqn (1) is straightforward if the system is proportionally damped, that is, if

$$[C] = \alpha[M] + \beta[K], \tag{2}$$

where α and β are arbitrary constants, or, more generally, if

$$[C][M]^{-1}[K] = [K][M]^{-1}[C]. \tag{3}$$

When eqn (3) is not satisfied, the system is said to be nonclassically damped.

In the following sections, we address the convergence of the series-based method described in Refs [3, 4]. Having observed that the convergence of the method as described in the references is not a straightforward matter, we noted that a subtle modification of the formulation leads to a simple convergence condition. We thus describe in the next section our modification to the method, and in the section following the next, we develop the criterion for the convergence of the revised series-based method.

In the work reported here, we limited ourselves to nonrepeated eigenvalues. We allow nonsymmetric damping matrices. In such cases the eigenvectors discussed here are properly termed "right eigenvectors". For brevity we shall, however, use the terms "eigenvector", and "eigenvectors" throughout. We note in passing that for nonsymmetric systems, left eigenvectors may also be developed by the method we describe. Our work on convergence applies to these vectors, too.

2.2. *Eigenanalysis*

Equation (1) is transformed by the change of variables,

$$\{x\} = [\Phi]\{y\}, \tag{4}$$

and a premultiplication by $[\Phi]^T$. The $(n \times n)$ matrix $[\Phi]$ has as its columns the eigenvectors of the undamped system, that is, of the system for which $[C] = [0]$. We assume that $[\Phi]$ is normalized such that,

$$[\Phi]^T[M][\Phi] = [I], \tag{5}$$

where $[I]$ is the identity matrix.

The result of the above operations is,

$$[I]\{\ddot{y}\} + [\Gamma]\{\dot{y}\} + [\Lambda]\{y\} = \{0\}. \tag{6}$$

In eqn (6),

$$[\Lambda] = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2). \tag{7}$$

Assuming a solution to eqn (6) of the form,

$$\{y\} = \{u\}e^{st}, \tag{8}$$

we obtain a quadratic eigenproblem,

$$(s^2[I] + s[\Gamma] + [\Lambda])\{u\} = \{0\}. \tag{9}$$

Equation (9) is solved in Refs [3, 4] after the transformed damping matrix, $[\Gamma]$, is written as,

$$[\Gamma] = [\Gamma_0] + \epsilon[\Gamma_1], \tag{10}$$

where $[\Gamma_0]$ is diagonal and where $[\Gamma_1]$ has zeros on the diagonal. For the jth eigenvalue and eigenvector, power series in ϵ are assumed. These are respectively:

$$s_j = \sum_{i=0}^{\infty} s_{ji}\epsilon^i \tag{11}$$

$$\{u_j\} = \sum_{i=0}^{\infty} \{u_{ji}\}\epsilon^i. \tag{12}$$

The substitution of eqns (10)–(12) into eqn (9), with observations and manipulations, led in the references cited to expressions for s_{ji} and $\{u_{ji}\}$, the ith correction to the jth eigenvalue and eigenvector, respectively.

For the reason to be discussed in the next section, we do not use the substitution given in eqn (10). Rather, we replace $[\Gamma]$ by $\epsilon[\Gamma]$, that is,

$$[\Gamma] \Rightarrow \epsilon[\Gamma]. \tag{13}$$

Substituting eqns (11)–(13) into eqn (9), we obtain after manipulations,

$$\sum_{i=0}^{\infty} \epsilon^i\{T_{ji}\} = \{0\}, \tag{14}$$

where

$$\{T_{j0}\} = [A_j]\{u_{j0}\} \tag{15}$$