$$\begin{pmatrix} x(0) \\ y(0) \\ u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y(x_0) \\ \left[x^2 D^2 + xD + (x^2 - m^2)\right] y(x_0) \\ xDy(x_0) \end{pmatrix}. \quad (22)$$

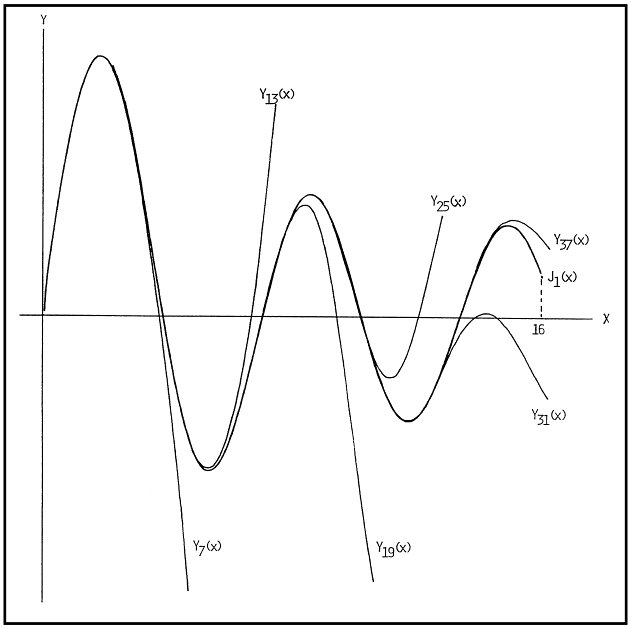

Figure 4

Simulating (22) with the analog computer, we used only four integrations and two multiplications of time dependent variables to illustrate polynomials of degree up to 37 as shown in Figure 4. ❑

[1] E.A. Coddington and N. Levenson, *Theory of Ordinary Differential Equations*, McGraw Hill, New-York, 1955.

[2] Wilfred Kaplan, *Ordinary Differential Equations*, Addison-Wesley, Reading, Mass., 1958.

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[4] T.A. Newton, "On using the analog computer to illustrate space curves," *Proc. Second Annual Houston Conference on Circuits, Systems, and Computers*, April 20–21, 1970, pp. 298–307.

[5] _____ , "On using a differential equation o generate polynomials," *The American Mathematical Monthly*, June–July 1974, 592–601.

[6] _____ , "Some parametric techniques in the analog solution of differential equations," *IEEE Transactions on Computers*, January 1975, 1–8.

Tyre A. Newton, Professor Emeritus of Pure and Applied Mathematics at Washington State University, has long had an interest in differential equations and computing. Though retired in 1985, Tyre is still doing mathematics; he is an inspiration to graduate students and faculty alike. His office is a veritable museum of late 20th century computing technology, complete with analog computer, an Apple II, and every Intel PC whose CPU identifier contains the numeral "8". In his spare time Tyre travels the country with his lovely wife Ellie, and is an expert fly-fisherman and outdoorsman.—M.E.M.

An Experimental Harvest from the Logistic Equation

Steve Clark, Scott Coble, Tim Randolph, Michael Moody

Introduction. The equation

$$n'(t) = rn(t)\left(1 - \frac{n(t)}{k}\right) \quad (1)$$

was first proposed by Verhulst in 1836 to describe a process in which the growth of a population, whose census at time t is $n(t)$, would be limited by its size; that is, should $n(t)$ become too large, the rate of change of $n(t)$ should then be small—perhaps even negative. Equation (1) is called the *logistic* differential equation, and growth described by this equation is said to be *logistic growth*. The constant r is said to be the *intrinsic growth rate* and the constant k is said to be the *carrying capacity* of the environment.

The logistic equation is the starting point for the development of many models in population biology. In this experiment we will explore a variation of the logistic model that incorporates *harvesting*. We begin with a few exercises to review the qualitative behavior of the logistic equation. The following exercises may be done with pencil and paper, or use your CAS as appropriate.

Exercise 1. Show that the constant functions $n(t) = 0$ and $n(t) = k$ for all $t \geq 0$ are solutions of the logistic equation. The first of these solutions, $n \equiv 0$, corresponds to a population that has gone extinct. What situation does the other constant solution describe? Why are these solutions called *equilibrium solutions* to the equation?

Exercise 2. Show that if $0 < n(t) < k$ then $n'(t) > 0$ (the population size is increasing), whereas if $n(t) > k$ then the population size decreases ($n'(t) < 0$).

Exercise 3. Using the result of the previous exercise, argue that $\lim_{t \to \infty} n(t) = k$ if $n(0) > 0$. Check your result by solving the logistic equation using separation of variables. Why does it make sense to call k the carrying capacity?

Exercise 4. You proved in the previous exercise that the population size approaches the carrying capacity k: $\lim_{t \to \infty} n(t) = k$. This result does not depend on the specific value of the growth rate r. What role does r play in the logistic equation? To help see the answer, take $k = 100$ and plot solutions using your computer to the IVP

$$n'(t) = rn(t)\left(1 - \frac{n(t)}{100}\right), \quad n(0) = 10 \qquad (2)$$

for $r = 0.5, 1, 2, 3$; compare the graphs of the solutions.

A modified equation—harvesting. The experiments that follow explore the effects of *harvesting* on an isolated species. Imagine that our population represents some resource, such as the fish in a lake or the amount of timber in a forest. The resource has some economic value, and so we humans wish to remove or harvest the resource. We will assume that in the absence of harvesting, the resource population obeys the logistic growth law with growth rate r and carrying capacity k.

To account for consumption of the resource population, we modify the logistic equation by subtracting a nonnegative function, $h(t)$, describing the rate at which the resource is harvested:

$$n'(t) = r\left(1 - \frac{n(t)}{k}\right)n(t) - h(t). \qquad (3)$$

The harvesting rate $h(t)$ could be specified as an explicit function of time, such as $h(t) = 1$, or implicitly through dependence on the size of the resource population, such as $h(t) = \rho n(t)$.

Exercise 1. Consider two harvest functions: $h_1(t) = \rho$ and $h_2(t) = \rho n(t)$, where ρ is a positive constant. Describe in words the qualitative difference between these two harvest functions. Which harvest function would most likely describe the harvesting of fish from a lake by sport fishermen, and which might describe the harvesting of a large hardwood forest?

Exercise 2. Explore the effect of a constant harvesting rate using h_1 and $\rho = 10$. Let $r = 0.3$ and $k = 1{,}000$. Use your computer to plot solutions for $0 \leq t \leq 100$ for values of $n(0)$ ranging from 10 to 1000 in increments of 100. Is it possible under these conditions to drive the resource population to extinction? How might this depend on the initial size of the resource population? Repeat your work, but now raise the harvesting rate to $\rho = 70$. Repeat again for $\rho = 100$. What effect does this increased harvesting have on the resource? Describe any "threshold effect" that you observe.

Exercise 3. If harvesting is at a constant rate ρ, show analytically that $n' < 0$ for all $n \geq 0$ if $\rho > kr/4$; thus the resource will always be exhausted if harvesting is too intense.

Exercise 4. Suppose that the constant harvesting rate ρ is not too large: $\rho > kr/4$. Now show that if the resource is initially sufficiently abundant when harvesting begins ($n(0) > n_1 > 0$), then the resource population converges to a sustainable equilibrium level (i.e., $n(t) \to n_2 > 0$, etc.), where

$$n_1 = \frac{k}{2}\left(1 - \sqrt{1 - \frac{4\rho}{kr}}\right), \quad n_2 = \frac{k}{2}\left(1 + \sqrt{1 - \frac{4\rho}{kr}}\right).$$

Show that if the initial population size is below n_1, then harvesting will exhaust the resource.

Exercise 5. Does it bother you that $n = 0$ is *not* an equilibrium solution for the logistic equation with constant harvesting? Discuss this apparent contradiction with reality; try to answer the question: when is this model likely to be reasonable, and when not.

Exercise 6. How does the situation change if we use $h_2 = \rho n$ for the harvesting rate? Does this model ever lead to biologically absurd population sizes? Can you find the "best" harvesting rate that sustains the resource?

Exercise 7. To prevent over exploitation of resources, conservation authorities often restrict harvesting (fishing, hunting etc.) to sharply defined "seasons"; no (legal) harvesting occurs out-of-season. We can model periodic harvesting using a "square wave" harvesting function. Suppose, for example, that harvesting occurs at the constant rate ρ for one unit of time and that there is no harvesting for the next three units of time, with the pattern repeating. We could model this harvesting pattern using the function $h(t) = \rho$ if $t \bmod 4 < 1$ and zero otherwise. Use your computer to plot a graph of this $h(t)$ for $\rho = 100$. A plot of this harvesting rate function appears in Figure 1.

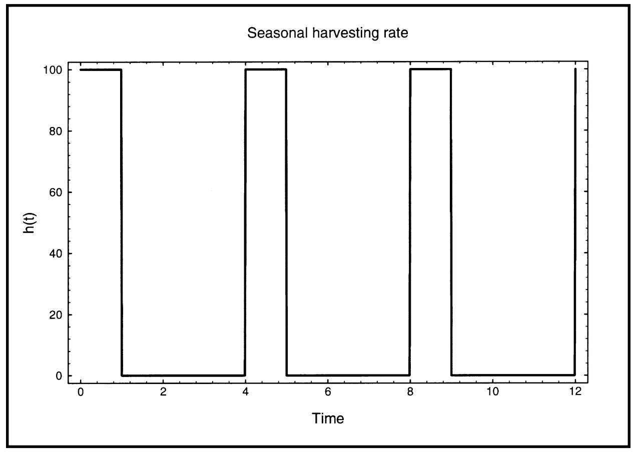

Figure 1

Suppose that the resource that we are harvesting is fish, and that population size is measured in some natural unit, say thousands of fish. Set $r = 0.3, k = 1000, \rho = 100$. Let $n(0) = 200$; with this value of $n(0)$ confirm that with constant harvesting and $\rho = 100$ the population goes extinct (see *Exercise 3*). Now plot the solution to the ODE using the seasonal harvest rate function and show that the population persists with seasonal harvesting. A plot of population size versus time for this situation appears in Figure 2. Can you explain the "jagged" appearance of the plot?

Exercise 8. Fix the period of the harvesting function at 4 time units. Now modify the periodic harvesting function to increase the duration of the harvesting season relative to the non-harvesting season. Try to discover, to a reasonable approximation, the maximum length of the harvesting season that will sustain a population that begins with 200 individuals.

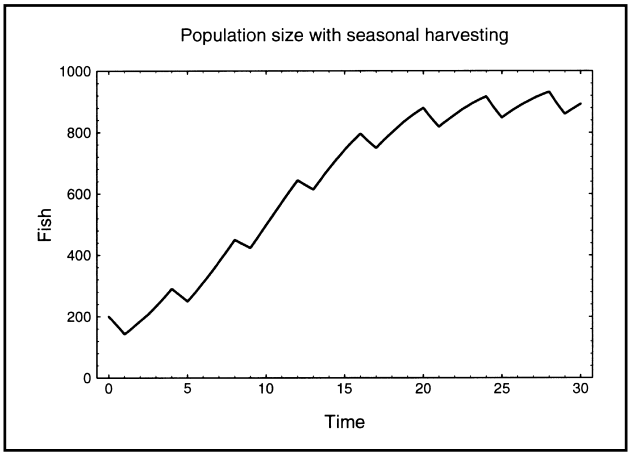

Figure 2

Conclusions and extensions. There are many models used by ecologists and wildlife biologists to describe the effects of harvesting on resource populations. Though many of these models are more complicated than those that we have investigated above, our simple model shares several important kinds of behavior with them. In particular, we discovered that over-exploitation can lead to ruin; that whether or not a given harvest rate is sustainable may depend on the initial state of a population; and that seasonal harvesting can be adjusted to accommodate sustainable exploitation of a resource.

A simple way to modify our harvest function to imitate a constant harvest rate at high population densities, and a density-dependent harvest rate at low population numbers would be to take

$$h = p\frac{n}{\eta + n},$$

where η is a new parameter that can be used to adjust the transition between the low-density and high-density regimes. If you are interested, explore the dynamics of the model using this harvest function. ❑

DIFFERENTIAL EQUATIONS AT MIDWESTERN STATE

Mark Ferris

The last time I taught the first course in differential equations I organized the material around a collection of particular equations rather than around a collection of solution techniques. Most of the class time was spent on student exploration using MDEP. I also included some ideas from dynamical systems that I hadn't mentioned in the past. These three conditions caused me to carefully weigh which analytic techniques and which mathematical models to include. The topics that were included are listed below. Hopefully this will add to the debate about what topics to include in a first year course.

MSU is a small state school in North Texas. We offer one semester of differential equations. A section typically has 15–20 students majoring predominantly in Mathematics, Chemistry, or Computer Science. In the Fall '93 semester the class met twice a week for 80-minute sessions in a laboratory equipped with 24 networked 386 machines. Students worked independently or in pairs. Students that worked in pairs often used two machines at a time. While MDEP was the primary tool, several students supplemented this with graphing calculators and occasionally a spreadsheet. The absence of symbolic manipulation software was primarily due to local financial constraints, but it is