Evolution semigroups and stability of time-varying systems on Banach spaces

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Abstract

The two main topics addressed are: (i) the relationship between internal, external, and input-output stability, and (ii) stability of time-invariant systems including a new Banach-space formula for the stability radius. With regard to (i), we show that a nonautonomous system is internally stable if and only if it is stabilizable, detectable and input-output stable; the short proof seems to be new even for finite-dimensional autonomous systems. For (ii), new formulas are given, in terms of the coefficients of the system, for the L_p-norm of the input-output operator and for the stability radius of the system.

1 Introduction

Using evolution semigroups we continue a study, begun in [8] of stability questions for infinite-dimensional linear time-varying state-space systems. Of the two main results, the first characterizes exponential stability of time-varying systems on Banach spaces in terms of the input-output operator, while the second provides a formula for the stability radius of time-invariant systems in terms of the L_p-norm of this operator. Both results extend, in a natural way, classical theorems for Hilbert-space systems. We stress that these classical statements do not generally hold for Banach-spaces or when $p \neq 2$.

Consider systems of the form

$$\begin{aligned} x(t) &= \Phi(t,s)x_s + \int_s^t \Phi(t,\tau)B(\tau)u(\tau)\,d\tau, \\ y(t) &= C(t)x(t), \qquad t \geq 0. \end{aligned} \tag{1.1}$$

Here $\Phi = \{\Phi(t,s)\}_{t\geq s}$ is a strongly continuous exponentially bounded evolution family (propagator), $B(\cdot) \in L_\infty(\mathbb{R}_+, B_s(U,X))$ and $C(\cdot) \in L_\infty(\mathbb{R}_+, B_s(X,Y))$, where X, Y, U are Banach spaces and $B_s(\cdot\,,\cdot)$ is the set of bounded operators with the topology of strong convergence. In particular, our results apply to systems modeled with time-varying differential equations on Banach spaces

$$x'(t) = A(t)x(t) + B(t)u(t),\ y(t) = C(t)x(t),\ t \geq 0, \tag{1.2}$$

arising from partial or functional differential equations where the operators $A(t)$ are not assumed to be bounded. For time-invariant (autonomous) systems, we consider a "mild" version of (1.2);

$$\begin{aligned} x(t) &= e^{tA}x_0 + \int_s^t e^{(t-\tau)A}Bu(\tau)\,d\tau, \\ y(t) &= Cx(t), \qquad t \geq 0, \end{aligned} \tag{1.3}$$

where $\{e^{tA}\}_{t\geq 0}$ is a strongly continuous semigroup on X generated by A, where the operators $B : U \to X$ and $C : X \to Y$ are bounded, and where A, B, and C are time independent.

For the general systems (1.1), define the input-output operator, $\mathbb{L}$, for $u \in L_p(\mathbb{R}_+; U)$, $1 \leq p < \infty$, as

$$\mathbb{L}u(t) = C(t)\int_0^t \Phi(t,\tau)B(\tau)u(\tau)\,d\tau, \quad t \geq 0,$$

and recall the following notions of stability. The system (1.1) is *input-output stable* if $\mathbb{L}$ is a bounded operator from $L_p(\mathbb{R}_+, U)$ to $L_p(\mathbb{R}_+, Y)$; this system is called *internally stable* if the evolution family Φ is exponentially stable, that is, $\|\Phi(t,s)\| \leq Me^{-\beta(t-s)}$ for some constants $\beta > 0$, $M > 0$ and all $t \geq s$. The autonomous system (1.3) is called *externally stable* if the transfer function $H(s) := C(A-s)^{-1}B$ is a bounded analytic function of s in the right half-plane $\mathbb{C}_0 = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$.

Since it is often desirable to deduce stability based solely on the knowledge of inputs and outputs, the relationship between internal stability and the other two types of stability has been examined fairly extensively. See, for example, [1] (finite-dimensional setting) and

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[13] (time-invariant, Hilbert-space setting) as well as [2, 5] and the references therein. Here, we extend a basic relationship, examined in these references, to the general setting of (1.1) (see Theorem 2.3 for a detailed statment):

Theorem 1.1 *System* (1.1) *is internally stable if and only if it is stabilizable, detectable and input-output stable.*

Consider now the time-invariant system (1.3). The following result of G. Weiss [14] holds for the Banach space setting:

$$\sup_{s\in\mathbb{C}_0} \|H(s)\| \leq \|\mathbb{L}\|_{B(L_p(\mathbb{R}_+,U),L_p(\mathbb{R}_+,Y))}. \qquad (1.4)$$

If U and Y are Hilbert spaces and $p=2$, then equality holds in (1.4) (see, e. g., [3] or [8]), and so Theorem 1.1 recovers a theorem by R. Rebarber [13], saying that *a Hilbert space autonomous system is internally stable if and only if it is stabilizable, detectable and externally stable.* Using known examples, we observe (Example 2.2 below) that for Banach spaces, the hypotheses of stabilizability and detectability are *not* sufficient to ensure that external stability implies internal stability. This state of affairs is hinted at by the fact that in Banach spaces a *strict* inequality can hold in (1.4). On the other hand, Theorem 3.1 below shows that the conclusion of the Rebarber's theorem does hold for the Banach-space setting under the additional assumption that the equality $s_0(A)=\omega_0(A)$ is satisfied; here, for a strongly continuous semigroup $\{e^{tA}\}_{t\geq0}$, $\omega_0(A)$ denotes the growth bound and $s_0(A)$ denotes the abscissa of uniform boundedness of the resolvent: $s_0(A)=\inf\{\omega\in\mathbb{R}:\mathbb{C}_\omega\subset\rho(A)$ and $\sup_{\lambda\in\mathbb{C}_\omega}\|(\lambda-A)^{-1}\|<\infty\}$, where $\mathbb{C}_\omega:=\{z\in\mathbb{C}:\mathrm{Re}(z)>\omega\}$ for $\omega\in\mathbb{R}$. In Corollary 3.2 we formulate a checkable sufficient condition from [6] (see also [10, Cor. 4.6.12]) under which this equality indeed holds.

It should be pointed out that [13] allows for a certain degree of unboundedness of the operators B and C. Such "regular" systems (see [14]), and their time-varying generalizations, might be addressed by combining the techniques of the present paper with those of [3] and [4]. This will not be done here.

The next theorem gives a Banach-space replacement for the Hilbert-space *equality* (1.4); that is, it provides a formula for the norm of the input-output operator in terms of the transfer function (see also Theorem 3.3 below). Here, $\mathcal{S}(\mathbb{R},U)$ denotes the Schwartz class of rapidly decreasing U-valued functions on $\mathbb{R}$.

Theorem 1.2 *Assume that the autonomous system* (1.3) *is internally stable. Then* $\mathbb{L}\in B(L_p(\mathbb{R},U), L_p(\mathbb{R},Y))$, *and* $\|\mathbb{L}\|$ *is equal to*

$$\sup_{u\in\mathcal{S}(\mathbb{R},U)} \frac{\|\int_{\mathbb{R}} C(A-is)^{-1}Bu(s)e^{is(\cdot)}\,ds\|_{L_p(\mathbb{R},Y)}}{\|\int_{\mathbb{R}} u(s)e^{is(\cdot)}\,ds\|_{L_p(\mathbb{R},U)}}. \qquad (1.5)$$

In view of the recent result in [11], this theorem provides a new expression for the stability radius, see [3], a quantity that describes a "distance from instability". More specifically, assume that the *time-invariant* system (1.3) is internally stable, and view B and C as operators describing the structure of a perturbation. If $\Delta(\cdot):\mathbb{R}_+\to B(Y,U)$ denotes a bounded, strongly measurable operator-valued function—viewed as an unknown disturbance—then the stability radius measures the size of the smallest Δ for which (mild) solutions to the perturbed equation,

$$x'(t)=(A+B\Delta(t)C)x(t) \qquad (1.6)$$

loose exponential stability. This is denoted by $r_{stab}(A,B,C)=\sup\{r>0:\|\Delta\|_\infty\leq r$ implies that (1.6) is exponentially stable$\}$. When $p=2$ and U and Y are Hilbert spaces, the equality

$$\|\mathbb{L}\|^{-1}=r_{stab}(A,B,C)=\left[\sup_{s\in\mathbb{R}}\|H(s)\|\right]^{-1} \qquad (1.7)$$

is known; see, e.g., [3]. In the general Banach-space setting the first and third quantities may differ [8]. However, using evolution semigroups as in [8, 9], J. van Neerven [11] has recently shown that the stability radius *equals* $1/\|\mathbb{L}\|$. Thus, the reciprocal of (1.5) gives a Banach-space formula for the stability radius in terms of the transfer function. As such, it generalizes the Hilbert-space formula (1.7) and makes a new connection between the state-space and frequency-domain objects, $\mathbb{L}$ and $H(s)$.

2 Stability

The techniques used in proving Theorems 1.1 and 1.2 rely on the concept of evolution semigroups and their spectral properties. The term *evolution semigroup* here refers to a family of operators $\{E^t\}_{t\geq0}$ on $L_p(\mathbb{R}_+,X)$ which is induced by a strongly continuous exponentially bounded evolution family Φ by the rule $(E^tf)(\tau)=\Phi(\tau,\tau-t)f(\tau-t)$ for $0\leq t\leq\tau$ and $(E^tf)(\tau)=0$ for $0\leq\tau<t$; its generator will be denoted by Γ.

It is known (see [8, 9, 10, 12] and the references therein), that the spectrum $\sigma(\Gamma)$ is invariant under translations along $i\mathbb{R}$, the spectrum $\sigma(E^t)$, is invariant under rotations about the origin, and $e^{t\sigma(\Gamma)}=\sigma(E^t)\setminus\{0\}$, $t>0$. Moreover, Φ is exponentially stable if and only if the semigroup $\{E^t\}_{t\geq 0}$ is stable, or, equivalently, the operator Γ has a bounded inverse on $L_p(\mathbb{R}_+, X)$. In addition, we have the following fact from [8] which is based on an autonomous version by **J.** van Neerven [9].

Theorem 2.1 *An evolution family of operators* Φ *on* X, *is exponentially stable if and only if*

$$\mathbb{G}f(t) = \int_0^t \Phi(t,\tau)f(\tau)\,d\tau, \quad t \geq 0, \tag{2.8}$$

defines a bounded operator, $\mathbb{G}$, *on* $L_p(\mathbb{R}_+, X)$*; in this case,* $\mathbb{G} = -\Gamma^{-1}$.

This characterization of stability allows Theorem 1.1 to be proven using elementary algebra of operators. Before giving this proof, we state the general definitions of stabilizability and detectability and point out an (autonomous) example showing that such conditions are not sufficient to ensure that external stability implies internal stability.

The system (1.1) is said to be *stabilizable* if there exists $F(\cdot) \in L_\infty(\mathbb{R}_+, B_s(X,U))$ and a corresponding exponentially stable evolution family, $\{\Phi_{BF}(t,s)\}_{t\geq s}$ such that, for $t \geq s$ and $x \in X$, one has $\Phi_{BF}(t,s)x = \Phi(t,s)x + \int_s^t \Phi(t,\tau)B(\tau)F(\tau)\Phi_{BF}(\tau,s)x\,d\tau$. The system (1.1) is said to be *detectable* if there exists $K(\cdot) \in L_\infty(\mathbb{R}_+, B_s(Y,X))$ and a corresponding exponentially stable evolution family, $\{\Phi_{KC}(t,s)\}_{t\geq s}$ such that, for $t \geq s$ and $x \in X$, one has $\Phi_{KC}(t,s)x = \Phi(t,s)x + \int_s^t \Phi_{KC}(t,\tau)K(\tau)C(\tau)\Phi(\tau,s)x\,d\tau$.

Example 2.2 Let A be a generator of a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$. It is always the case that $s_0(A) \leq \omega_0(A)$, see, e. g., [10]. Consider an A with the *strict* inequality $s_0(A) < \omega_0(A)$. To be more specific, rescaling an example due to W. Arendt (see [10], Example 1.4.5) gives a (positive) strongly continuous semigroup with a generator A having the property that $s_0(A) < \omega_0(A) = 0$. With this particular A, and $B = I$ and $C = I$, the system (1.3) is stabilizable (take $F = -\alpha I$ where $\alpha > 0$, and so $\omega_0(A + BF) = \omega_0(A) - \alpha < 0$), and detectable. Also, $s_0(A) < 0$, and so $H(s) = (s - A)^{-1}$ is a bounded function of s on $\mathbb{C}_0$ by the definition of $s_0(A)$. Since $\omega_0(A) = 0$, this system is not internally stable. $\diamond$

For the detailed version of Theorem (1.1) which follows, we define multiplication operators $\mathcal{B}$ and $\mathcal{C}$ by $\mathcal{B}u(t) = B(t)u(t)$, $u \in L_p(\mathbb{R}_+, U)$ and $\mathcal{C}f(t) = C(t)f(t)$, $f \in L_p(\mathbb{R}_+, X)$; if $B(\cdot)$ and $C(\cdot)$ are as in (1.1), then $\mathcal{B} \in B(L_p(\mathbb{R}_+, U), L_p(\mathbb{R}_+, X))$ and $\mathcal{C} \in B(L_p(\mathbb{R}_+, X), L_p(\mathbb{R}_+, Y))$.

Theorem 2.3 *The following are equivalent :*

i) $\{\Phi(t,s)\}_{t\geq s}$ *is exponentially stable on* X*;*

ii) $\mathbb{G}$ *is a bounded operator on* $L_p(\mathbb{R}_+, X)$*;*

iii) system (1.1) *is stabilizable and* $\mathbb{G}\mathcal{B}$ *is a bounded operator from* $L_p(\mathbb{R}_+, U)$ *to* $L_p(\mathbb{R}_+, X)$*;*

iv) system (1.1) *is detectable and* $\mathcal{C}\mathbb{G}$ *is a bounded operator from* $L_p(\mathbb{R}_+, X)$ *to* $L_p(\mathbb{R}_+, Y)$*;*

v) system (1.1) *is stabilizable and detectable and* $\mathbb{L} = \mathcal{C}\mathbb{G}\mathcal{B}$ *is a bounded operator from* $L_p(\mathbb{R}_+, U)$ *to* $L_p(\mathbb{R}_+, Y)$.

Proof: Assume that $\mathbb{L} = \mathcal{C}\mathbb{G}\mathcal{B}$ is a bounded operator. The definition of detectability gives $\Phi_{KC}(t,s)Bu = \Phi(t,s)Bu + \int_s^t \Phi_{KC}(t,\tau)\ K(\tau)C(\tau)\Phi(\tau,s)Bu\,d\tau$ for some $K(\cdot) \in L_\infty(\mathbb{R}_+, B_s(Y,X))$, and $\{\Phi_{KC}(t,s)\}_{t\geq s}$. Since the latter is exponentially stable, the operator $\mathbb{G}_{KC}f(t) := \int_0^t \Phi_{KC}(t,\tau)f(\tau)\,d\tau$ is a bounded operator on $L_p(\mathbb{R}_+, X)$. Integrating from 0 to t leads to $\mathbb{G}_{KC}\mathcal{B}u = \mathbb{G}\mathcal{B}u + \mathbb{G}_{KC}\mathcal{K}\mathcal{C}\mathbb{G}\mathcal{B}u$. Since $\mathcal{C}\mathbb{G}\mathcal{B}$, and $\mathbb{G}_{KC}$ are bounded, the operator $\mathbb{G}\mathcal{B}$ is bounded. Using the hypothesis of stabilizability, a similar calculation results in the equation $\mathbb{G}_{BF} = \mathbb{G} + \mathbb{G}\mathcal{B}\mathcal{F}\mathbb{G}_{BF}$, where $\mathbb{G}_{BF}f(t) = \int_0^t \Phi_{BF}(t,\tau)f(\tau)\,d\tau$ defines a bounded operator since $\{\Phi_{BF}(t,s)\}_{t\geq s}$ is exponentially stable. Hence, $\mathbb{G}$ is bounded. By Theorem 2.1, $\{\Phi(t,s)\}_{t\geq s}$ is then exponentially stable. ∎

3 The autonomous case

The main result of this section is Theorem 3.3, a detailed version of Theorem 1.2. This theorem parallels Theorem 2.3 and the main point is to provide explicit conditions, in terms of the operators A, B and C, which imply internal stability. In particular, the equivalence of (1) and (6) in Theorem 3.3 gives a direct Banach space generalization of the standard Hilbert space characterization of internal stability via the boundedness of the transfer function in the right half-plane.

Before proceeding with this theorem, recall (see Example 2.2) that the properties of stabilizability and detectability are not sufficient to ensure that external stability implies internal stability. As seen by the following Theorem 3.1, this failure is a consequence of the fact that $s_0(A)$, the abscissa of uniform boundedness, can be strictly less than the growth bound, $\omega_0(A)$.

Theorem 3.1 *Let* $\{e^{tA}\}_{t\geq 0}$ *be a* C_0 *semigroup with the property that* $s_0(A) = \omega_0(A)$. *Assume* (1.3) *is stabilizable and detectable. If* $\mathbb{C}_0 \subseteq \rho(A)$ *and* $M := \sup_{s\in\mathbb{R}} \|C(A - is)^{-1}B\| < \infty$, *then* $\{e^{tA}\}_{t\geq 0}$ *is exponentially stable.*

Proof: Choose operators $F \in B(X,U)$ and $K \in B(Y,X)$ such that the semigroups generated by $A+BF$ and $A+KC$ are exponentially stable, so it follows that $s_0(A+BF) < 0$ and $s_0(A+KC) < 0$. Set $M_1 = \sup_{s\in\mathbb{R}} \|(A+BF-is)^{-1}\|$ and $M_2 = \sup_{s\in\mathbb{R}} \|(A+KC-is)^{-1}\|$. Since $(A-is)^{-1}B = (A+KC-is)^{-1}B + (A+KC-is)^{-1}KC(A-is)^{-1}B$, it follows that

$$M_3 := \sup_{s\in\mathbb{R}} \|(A-is)^{-1}B\| \le M_2\|B\| + M_2\|K\|M.$$

Also, $(A-is)^{-1} = (A+BF-is)^{-1} + (A-is)^{-1}BF(A+BF-is)^{-1}$, and so $\sup_{s\in\mathbb{R}} \|(A-is)^{-1}\| \le M_1 + M_3\|F\|M_1$. Therefore, $\omega_0(A) = s_0(A) < 0$. ∎

Corollary 3.2 *Assume there exists an $\omega > \omega_0(A)$ such that, for each $x \in X$ and each $x^* \in X^*$, the adjoint space, one has $\int_{\mathbb{R}} \|(\omega+i\tau-A)^{-1}x\|_X^2\, d\tau < \infty$ and $\int_{\mathbb{R}} \|(\omega+i\tau-A^*)^{-1}x^*\|_{X^*}^2\, d\tau < \infty$. Then (1.3) is internally stable if and only if it is stabilizable, detectable and externally stable.*

Proof: According to [6] (see also [10, Cor. 4.6.12]), the conditions above imply $s_0(A) = \omega_0(A)$. Now Theorem 3.1 gives the result. ∎

We now state the main theorem of the section. Let $A_\alpha := A - \alpha I$ denote the generator of the rescaled semigroup $\{e^{-\alpha t}e^{tA}\}_{t\ge0}$.

Theorem 3.3 *The following are equivalent for a strongly continuous semigroup $\{e^{tA}\}_{t\ge0}$ generated by A on a Banach space:*

(1) $\{e^{tA}\}_{t\ge0}$ *is exponentially stable;*

(2) $\mathbb{G}$ *is a bounded operator on $L_p(\mathbb{R}_+,X)$;*

(3) $$\sup_{v\in\mathcal{S}(\mathbb{R},X)} \frac{\|\int_{\mathbb{R}}(A_\alpha - is)^{-1}v(s)e^{is(\cdot)}\,ds\|_{L_p(\mathbb{R},X)}}{\|\int_{\mathbb{R}} v(s)e^{is(\cdot)}\,ds\|_{L_p(\mathbb{R},X)}}$$
is finite for all $\alpha \ge 0$;

(4) $$\sup_{u\in\mathcal{S}(\mathbb{R},U)} \frac{\|\int_{\mathbb{R}}(A_\alpha - is)^{-1}Bu(s)e^{is(\cdot)}\,ds\|_{L_p(\mathbb{R},X)}}{\|\int_{\mathbb{R}} u(s)e^{is(\cdot)}\,ds\|_{L_p(\mathbb{R},U)}}$$
is finite for all $\alpha \ge 0$ and (1.3) is stabilizable;

(5) $$\sup_{v\in\mathcal{S}(\mathbb{R},X)} \frac{\|\int_{\mathbb{R}}C(A_\alpha - is)^{-1}v(s)e^{is(\cdot)}\,ds\|_{L_p(\mathbb{R},Y)}}{\|\int_{\mathbb{R}} v(s)e^{is(\cdot)}\,ds\|_{L_p(\mathbb{R},X)}}$$
is finite for all $\alpha \ge 0$ and (1.3) is detectable;

(6) $$\sup_{u\in\mathcal{S}(\mathbb{R},U)} \frac{\|\int_{\mathbb{R}}C(A_\alpha - is)^{-1}Bu(s)e^{is(\cdot)}\,ds\|_{L_p(\mathbb{R},Y)}}{\|\int_{\mathbb{R}} u(s)e^{is(\cdot)}\,ds\|_{L_p(\mathbb{R},U)}}$$
is finite for all $\alpha \ge 0$ and (1.3) is both stabilizable and detectable.

Moreover, if $\{e^{tA}\}$ is exponentially stable, then the norm of the input-output operator, $\mathbb{L} = \mathcal{C}\mathbb{G}\mathcal{B}$, is

$$\sup_{u\in\mathcal{S}(\mathbb{R},U)} \frac{\|\int_{\mathbb{R}}C(A_\alpha - is)^{-1}Bu(s)e^{is(\cdot)}\,ds\|_{L_p(\mathbb{R},Y)}}{\|\int_{\mathbb{R}} u(s)e^{is(\cdot)}\,ds\|_{L_p(\mathbb{R},U)}}.$$

As noted in (1.7), If U and Y are Hilbert spaces and $p=2$, then $\|\mathbb{L}\| = \sup_{s\in\mathbb{R}} \|C(A-is)^{-1}B\|_{B(U,Y)}$.

Proof: The proof proceeds by first showing that statements (1)–(3) are equivalent. This is done by relating properties of the evolution semigroup $\{E^t\}_{t\ge0}$ on $L_p(\mathbb{R}_+,X)$ to the properties of an evolution semigroup of operators acting on $L_p(\mathbb{R},X)$. This argument also proves the last statement (see Theorem 3.7) as well as the implication (1)⇒(6). The proof of Theorem 3.3 is completed by proving the implications (6)⇒(4)⇒(3) and (6)⇒(5)⇒(3).

We begin by considering an evolution semigroup on the entire real axis: for an evolution family $\{\Phi(t,s)\}_{t\ge s}$, define a semigroup $\{E^t_{\mathbb{R}}\}_{t\ge0}$ on $L_p(\mathbb{R},X)$ by $(E^t_{\mathbb{R}}f)(\tau) = \Phi(\tau,\tau-t)f(\tau-t)$, $\tau\in\mathbb{R}$, and denote its generator by $\Gamma_{\mathbb{R}}$. The current focus is on autonomous systems, so $E^t_{\mathbb{R}}$ takes the form $(E^t_{\mathbb{R}}f)(\tau) = e^{tA}f(\tau-t)$. The generator, $\Gamma_{\mathbb{R}}$, is given by the closure (in $L_p(\mathbb{R},X)$) of the operator $-d/dt + \mathcal{A}$ where $(\mathcal{A}f)(t) = Af(t)$, and $\mathcal{D}(-d/dt+\mathcal{A}) = \mathcal{D}(-d/dt) \cap \mathcal{D}(\mathcal{A}) = \{v \in L_p(\mathbb{R},X) : v \in AC(\mathbb{R},X),\ \ v' \in L_p(\mathbb{R},X),\ \ v(s) \in \mathcal{D}(\mathcal{A})$ for a. a. s, and $v' + Av \in L_p(\mathbb{R},X)\}$.

As shown in [7], the existence of an exponential dichotomy for solutions to $x'(t) = Ax(t)$, $t\in\mathbb{R}$, is characterized by the condition that $\Gamma_{\mathbb{R}}$ is invertible on $L_p(\mathbb{R},X)$. On the other hand, exponential stability of solutions to $x'(t) = Ax(t)$, $t\ge0$, is characterized by the condition that Γ is invertible on $L_p(\mathbb{R}_+,X)$ [8]. We shall begin by assuming that (1) holds. In this case both Γ and $\Gamma_{\mathbb{R}}$ are invertible.

Given $v \in \mathcal{S}(\mathbb{R},X)$, let g_v and f_v denote the functions $g_v(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} v(s)e^{i\tau s}\,ds$ and $f_v(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}}(A-is)^{-1}v(s)e^{i\tau s}\,ds$, and define the sets $\mathfrak{G} = \{g_v : v \in \mathcal{S}(\mathbb{R},X)\}$ and $\mathfrak{F} = \{f_v : v\in\mathcal{S}(\mathbb{R},X)\}$.

Proposition 3.4 *The set $\mathfrak{G}$ consists of differentiable functions and is dense in $L_p(\mathbb{R},X)$; the set $\mathfrak{F}$ is dense in $\mathcal{D}(\Gamma_{\mathbb{R}})$; for $v\in\mathcal{S}(\mathbb{R},X)$ one has $\Gamma_{\mathbb{R}}f_v = g_v$.*

Proof: Note that $\mathfrak{G} = \{g : \mathbb{R}\to X :$ there exists $v \in \mathcal{S}(\mathbb{R},X)$ so that $\hat{g} = v\}$, where $\hat{}$ is the Fourier transform, and so $\mathfrak{G}$ contains the set $\{g \in L^1(\mathbb{R},X) : \hat{g}\in\mathcal{S}(\mathbb{R},X)\}$, which is a dense in $L_p(\mathbb{R},X)$. The rest is a direct calculation. ∎

For an operator R on a Banach space Z we denote

$\|R\|_\bullet = \|R\|_{\bullet,Z} := \inf\{\|Rz\| : z \in \mathcal{D}(R), \|z\| = 1\}$. Also, set $\Lambda = \{v \in S(\mathbb{R}, X) : v(s) \in \mathcal{D}(A) \text{ for } s \in \mathbb{R}\}$.

Corollary 3.5 *Let Γ and $\Gamma_{\mathbb{R}}$, respectively, be the generators of the evolution semigroups on $L_p(\mathbb{R}_+, X)$ and $L_p(\mathbb{R}, X)$, as defined above. Assume $\sigma(A) \cap i\mathbb{R} = \emptyset$. Then the following assertions hold:*

i) $\|\Gamma_{\mathbb{R}}\|_{\bullet, L_p(\mathbb{R},X)}$ is equal to

$$\inf_{v\in\Lambda} \frac{\| \int_{\mathbb{R}} (A - is)v(s)e^{is(\cdot)}\, ds\|_{L_p(\mathbb{R},X)}}{\| \int_{\mathbb{R}} v(s)e^{is(\cdot)}\, ds\|_{L_p(\mathbb{R},X)}};$$

ii) if $\Gamma_{\mathbb{R}}$ is invertible on $L_p(\mathbb{R}, X)$, then $\{e^{tA}\}$ has exponential dichotomy and $\|\Gamma_{\mathbb{R}}^{-1}\|_{B(L_p(\mathbb{R},X))}$ is equal to

$$\sup_{v\in\mathcal{S}} \frac{\| \int_{\mathbb{R}} (A - is)^{-1}v(s)e^{is(\cdot)}\, ds\|_{L_p(\mathbb{R},X)}}{\| \int_{\mathbb{R}} v(s)e^{is(\cdot)}\, ds\|_{L_p(\mathbb{R},X)}};$$

iii) if Γ is invertible on $L_p(\mathbb{R}_+, X)$, then $\{e^{tA}\}$ is exponentially stable and $\|\Gamma^{-1}\|_{B(L_p(\mathbb{R}_+,X))} = \|\Gamma_{\mathbb{R}}^{-1}\|_{B(L_p(\mathbb{R},X))}$.

Proof: To show $i)$ let $v \in \mathcal{S}(\mathbb{R}, X)$. Then $w(s) = (A - is)^{-1}v(s)$, $s \in \mathbb{R}$, defines a function, w, in Λ. Now, $g_v(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} (A - is)w(s)e^{is\tau}\, ds$, and $f_v(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} w(s)e^{is\tau}\, ds$. However, from Proposition 3.4,

$$\begin{aligned}\|\Gamma_{\mathbb{R}}\|_\bullet &= \inf_{f_v\in\mathfrak{F}} \frac{\|\Gamma_{\mathbb{R}} f_v\|}{\|f_v\|} = \inf_{v\in\mathcal{S}} \frac{\|g_v\|}{\|f_v\|} \\ &= \inf_{w\in\Lambda} \frac{\| \int_{\mathbb{R}} (A - is)w(s)e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} w(s)e^{is(\cdot)}\, ds\|}.\end{aligned}$$

To see $ii)$ note that

$$\|\Gamma_{\mathbb{R}}^{-1}\| = \|\Gamma_{\mathbb{R}}\|_\bullet^{-1} = \left[\inf_{v\in\mathcal{S}} \frac{\|\Gamma_{\mathbb{R}} f_v\|}{\|f_v\|}\right]^{-1} = \sup_{v\in\mathcal{S}} \frac{\|f_v\|}{\|g_v\|}.$$

For $iii)$ note that $\|\Gamma_{\mathbb{R}}\|_{\bullet, L_p(\mathbb{R},X)} \le \|\Gamma\|_{\bullet, L_p(\mathbb{R}_+,X)}$ is trivial. To see that $\|\Gamma_{\mathbb{R}}\|_\bullet \ge \|\Gamma\|_\bullet$, let $\epsilon > 0$ and choose $f \in \mathcal{D}(\Gamma_{\mathbb{R}})$ with compact support such that $\|f\|_{L_p(\mathbb{R},X)} = 1$ and $\|\Gamma_{\mathbb{R}}\|_\bullet \ge \|\Gamma_{\mathbb{R}} f\| - \epsilon$. Now choose $\tau \in \mathbb{R}$ such that $f_\tau(s) := f(s - \tau)$, $s \in \mathbb{R}$, defines a function, $f_\tau \in L_p(\mathbb{R}, X)$, with $\operatorname{supp} f_\tau \subseteq \mathbb{R}_+$. Then $\|\Gamma_{\mathbb{R}}\|_\bullet \ge \|\Gamma_{\mathbb{R}} f\| - \epsilon = \|(\Gamma_{\mathbb{R}} f)_\tau\| - \epsilon \ge \|\Gamma\|_\bullet - \epsilon$. ■

Proposition 3.6 *The set $\mathfrak{G}_U = \{g_u : u \in \mathcal{S}(\mathbb{R}, U)\}$ is dense in $L_p(\mathbb{R}, U)$. If $u \in \mathcal{S}(\mathbb{R}, U)$ and $B \in B(U, X)$ then $Bu \in \mathcal{S}(\mathbb{R}, X)$ and $\Gamma_{\mathbb{R}} f_{Bu} = Bg_u$.*

Theorem 3.7 *If $\Gamma_{\mathbb{R}}$ is invertible on $L_p(\mathbb{R}, X)$, then $\|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\|$ is equal to*

$$\sup_{u\in\mathcal{S}(\mathbb{R},U)} \frac{\| \int_{\mathbb{R}} C(A_\alpha - is)^{-1}Bu(s)e^{is(\cdot)}\, ds\|_{L_p(\mathbb{R},Y)}}{\| \int_{\mathbb{R}} u(s)e^{is(\cdot)}\, ds\|_{L_p(\mathbb{R},U)}}. \tag{3.9}$$

If Γ is invertible on $L_p(\mathbb{R}_+, X)$, then, for $\mathbb{L} = \mathcal{C}\Gamma^{-1}\mathcal{B}$,

$$\|\mathbb{L}\|_{B(L_p(\mathbb{R}_+,X),L_p(\mathbb{R}_+,Y))} = \|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\|. \tag{3.10}$$

Proof: For $u \in \mathcal{S}(\mathbb{R}, U)$, consider functions f_{Bu} and g_u. Proposition 3.6 gives $f_{Bu} = \Gamma_{\mathbb{R}}^{-1}\mathcal{B}g_u$ and

$$\|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\| = \sup_{g_u\in\mathfrak{G}_u} \frac{\|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}g_u\|_{Lp(\mathbb{R},Y)}}{\|g_u\|_{Lp(\mathbb{R},U)}} = \sup_{g_u\in\mathfrak{G}_u} \frac{\|\mathcal{C}f_{Bu}\|}{\|g_u\|}$$

which is (3.9).

Now, if Γ is invertible on $L_p(\mathbb{R}_+, X)$, then $\{e^{tA}\}$ is exponentially stable [8, 9]. Hence, $\Gamma_{\mathbb{R}}$ is invertible on $L_p(\mathbb{R}, X)$. Moreover, for the case of the *stable* semigroup $\{e^{tA}\}_{t\ge0}$, the formula for $\Gamma_{\mathbb{R}}^{-1}$, for $f \in L_p(\mathbb{R}, X)$ with $\operatorname{supp} f \subseteq (0, \infty)$, takes the form

$$(\Gamma_{\mathbb{R}}^{-1}f)(t) = \int_{-\infty}^{t} e^{(t-s)A}f(s)\, ds = \int_{0}^{t} e^{(t-s)A}f(s)\, ds; \tag{3.11}$$

see, e. g., [7]. For a function $h \in L_p(\mathbb{R}_+, X)$, define an extension $\tilde{h} \in L_p(\mathbb{R}, X)$ by $\tilde{h}(t) = h(t)$ for $t \ge 0$ and $\tilde{h}(t) = 0$ for $t < 0$. Then (3.11) shows that $\Gamma_{\mathbb{R}}^{-1}\tilde{h} = (\Gamma^{-1}h)^\sim$. In particular, for $u \in L_p(\mathbb{R}_+, X)$ one has $\widetilde{\mathbb{L}u} = \widetilde{\mathcal{C}\Gamma^{-1}\mathcal{B}u} = \mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\tilde{u}$. Therefore, $\|\mathbb{L}\| \le \|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\|$, since

$$\|\mathbb{L}u\|_{L_p(\mathbb{R}_+,Y)} = \|\widetilde{\mathbb{L}u}\|_{Lp(\mathbb{R},Y)} \le \|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\| \cdot \|u\|_{L_p(\mathbb{R}_+,U)}.$$

To prove that equality holds in (3.10), let $\epsilon > 0$ and choose a compactly supported $u \in L_p(\mathbb{R}, U)$, $\|u\| = 1$, such that $\|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}u\|_{Lp(\mathbb{R},Y)} \ge \|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\| - \epsilon$. Now choose r such that $\operatorname{supp} u(\cdot - r) \subseteq (0, \infty)$ and set $w(\cdot) := u(\cdot - r)$. Then $w \in L_p(\mathbb{R}, U)$ with $\operatorname{supp} w \subseteq (0, \infty)$. Let $\bar{w}$ denote the element of $L_p(\mathbb{R}_+, U)$ that coincides with w on $\mathbb{R}_+$. As in (3.11) we have

$$\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}w(t) = C\int_{-\infty}^{t} e^{(t-s)A}Bw(s)\, ds,$$

which implies that

$$\begin{aligned}\|\mathbb{L}\| &\ge \|\mathbb{L}\bar{w}\|_{L_p(\mathbb{R}_+,Y)} = \|\widetilde{\mathbb{L}\bar{w}}\|_{Lp(\mathbb{R},Y)} = \|\mathbb{L}\tilde{\bar{w}}\|_{Lp(\mathbb{R},Y)} \\ &= \|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}w\| = \|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}u\| \ge \|\mathcal{C}\Gamma_{\mathbb{R}}^{-1}\mathcal{B}\| - \epsilon.\end{aligned}$$

■

To complete the proof of Theorem 3.3 we note that (2) follows from (1) by [9]. By Theorem 2.1, (1) is a consequence of (2). Exponential stability of $\{e^{tA}\}_{t\ge0}$

is equivalent to the invertibility of $\Gamma_{\mathbb{R}}$, and as a consequence, (3) follows from (1) by Corollary 3.5.

To show that (3) implies (1) we consider the case $\alpha = 0$ and assume that the expression in (3) is finite. By Proposition 3.4 and Corollary 3.5,

$$\|\Gamma_{\mathbb{R}}\|_{\bullet} = \inf_{u\in\mathcal{S}(\mathbb{R},X)} \frac{\|\Gamma_{\mathbb{R}} f_v\|}{\|f_v\|} = \sup_{v\in\mathcal{S}(\mathbb{R},X)} \left(\frac{\|f_v\|}{\|g_v\|}\right)^{-1} > 0.$$

This shows that $0 \notin \sigma_{ap}(\Gamma_{\mathbb{R}})$ and so (see, e.g., [7] or [12]), it follows that $\sigma_{ap}(e^{tA}) \cap \mathbb{T} = \phi$. On the other hand, since $\sigma(A) \cap i\mathbb{R} = \phi$, it follows from the Spectral Mapping Theorem for $\sigma_r(e^{tA})$ that $\sigma(e^{tA}) \cap \mathbb{T} = \left[\sigma_{ap}(e^{tA}) \cup \sigma_r(e^{tA})\right] \cap \mathbb{T} = \emptyset$. The same argument holds for any $\alpha \geq 0$. As a result, $\{e^{tA_\alpha}\}_{t\geq 0}$ is hyperbolic for each $\alpha \geq 0$, and thus $\omega_0(A) < 0$.

To see that (1) implies (6) note that if (1) holds then (1.3) is trivially exponentially stabilizable and detectable, and that $\{e^{tA\alpha}\}_{t\geq 0}$ is exponentially stable for all $\alpha > 0$; thus (6) follows from Theorem 3.7. The last statement of Theorem 3.3 also follows from Theorem 3.7.

To show that (6) implies (4), we begin by assuming that $\alpha = 0$. Since (1.3) is detectable, there exists $K \in B(Y, X)$ such that $A + KC$ generates an exponentially stable semigroup. By the equivalence of (1) and (3),

$$M_1 := \sup_{v\in\mathcal{S}(\mathbb{R},X)} \frac{\| \int_{\mathbb{R}} (A + KC - is)^{-1} v(s) e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} v(s) e^{is(\cdot)}\, ds\|}$$

is finite, and so,

$$\sup_{v\in\mathcal{S}(\mathbb{R},U)} \frac{\| \int_{\mathbb{R}} (A + KC - is)^{-1} Bu(s) e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} u(s) e^{is(\cdot)}\, ds\|}$$

is bounded by $M_1\|B\|$. By hypothesis,

$$M_2 := \sup_{u\in\mathcal{S}(\mathbb{R},U)} \frac{\| \int_{\mathbb{R}} C(A - is)^{-1} Bu(s) e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} u(s) e^{is(\cdot)}\, ds\|} < \infty.$$

For $u \in \mathcal{S}(\mathbb{R}, U)$, let $w(s) = KC(A - is)^{-1} Bu(s)$, $s \in \mathbb{R}$. Then,

$$\frac{\| \int_{\mathbb{R}} (A + KC - is)^{-1} KC(A - is)^{-1} Bu(s) e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} u(s) e^{is(\cdot)}\, ds\|}$$

is bounded by $M_1\|K\|M_2$. Finally, since $(A - is)^{-1}B = (A + KC - is)^{-1}B + (A + KC - is)^{-1}KC(A - is)^{-1}B$, it follows that

$$\sup_{u\in\mathcal{S}(\mathbb{R},U)} \frac{\| \int_{\mathbb{R}} (A - is)^{-1} Bu(s) e^{is(\cdot)}\, ds\|}{\| \int_{\mathbb{R}} u(s) e^{is(\cdot)}\, ds\|}$$

is bounded by $M_1\|B\| + M_1M_2\|K\|$ so (6) implies (4).

Arguments similar to these show that (4) implies (3), (6) implies (5) and that (5) implies (3). ■

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