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DICHOTOMY OF DIFFERENTIAL EQUATIONS ON BANACH SPACES AND AN ALGEBRA OF WEIGHTED TRANSLATION OPERATORS

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A strongly continuous evolutionary family $\{U(\tau,s)\}_{\tau\geq s}$ of operators on an arbitrary Banach space X is considered. This family can be viewed as a propagator for the differential equation $x'(\tau) = A(\tau)x(\tau)$, $\tau \in \mathbb{R}$, on X with, generally, unbounded operators $A(\tau)$. Associated with this family is an evolutionary semigroup $(e^{t\Gamma}f)(\tau) = U(\tau,\tau-t)f(\tau-t)$, $\tau \in \mathbb{R}$, on $L_p(\mathbb{R};X)$ with generator Γ which is often expressible by the formula $(\Gamma f)(\tau) = -f'(\tau) + A(\tau)f(\tau)$. We prove that the evolutionary family has an exponential dichotomy if and only if this semigroup is hyperbolic or, equivalently, its generator Γ is invertible.

To this end the invertibility of the operators from an algebra of weighted translation operators with strongly continuous coefficients on $L_p(\mathbb{R};X)$ is studied. We represent this algebra by weighted shift operators on $l_p(\mathbb{Z};X)$ and show, in addition, that this algebra is inverse-closed in $B(L_p(\mathbb{R};X))$. Consequences include a description of the Green's function for the evolutionary family in terms of Γ^{-1}. We show that the Green's function exists if and only if the evolutionary family has an exponential dichotomy. It is also shown that exponential dichotomy persists under small perturbations of the evolutionary family.

0. INTRODUCTION

In recent years, the classical ideas of [11,31] on exponential dichotomy and other asymptotic properties concerning the solutions of differential equations have witnessed significant development. A. Ben-Artzi, I. Gohberg, and M. Kaashoek in their recent papers [6,9] related the asymptotic behavior of the solutions of the matrix differential equation

$$x'(\tau) = A(\tau)x(\tau), \quad \tau \in \mathbb{R}, \tag{0-1}$$

on n-dimensional space $X = \mathbb{R}^n$ and the spectral properties of a differential operator Γ, defined on $L_p(\mathbb{R};X)$, $p \geq 1$, by the rule

$$(\Gamma f)(\tau) = -\frac{d}{d\tau}f(\tau) + A(\tau)f(\tau). \tag{0-2}$$

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In particular, it has been proven in [6] that the equation (0-1) has exponential dichotomy on $\mathbb{R}$ if and only if Γ is invertible on $L_p(\mathbb{R}; X)$, provided the function $\tau \mapsto \|A(\tau)\|$ is bounded on $\mathbb{R}$.

One of the goals of the present paper is to extend this result for the case in which X is an arbitrary Banach space and the operators $A(\tau)$ are, in general, unbounded. This extension is then used to describe the Green's function for a dichotomic equation (0-1) on X and to prove the "roughness" of exponential dichotomy. Our main technical tool is to consider an algebra of weighted translation operators on $L_p(\mathbb{R}; X)$ related to (0-2). Another purpose of this paper is to study the invertibility of the elements in this algebra, and to construct a representation of this algebra by the weighted shift operators on $\ell_p(\mathbb{Z}; X)$.

Our strategy is to begin not with the differential equation (0-1) as in [11,31], but instead to start from the semigroup of operators on $L_p(\mathbb{R}; X)$; this semigroup is often generated by an operator of the type (0-2). Then, from two facts about this semigroup (the spectral mapping theorem and the description of its spectral projections), one can obtain "for free" extensions of the classical results [11,31] concerning the asymptotic behavior of (0-1).

We consider any well-posed [14] differential equation (0-1) on a Banach space X. This means that instead of (0-1), we consider an *evolutionary family* of operators, $\{U(\tau, s)\}_{\tau \geq s}$, which can be viewed as the propagator for (0-1): $x(\tau) = U(\tau, s)x(s)$, $\tau \geq s$. We assume that $\{U(\tau, s)\}_{\tau \geq s}$ is a jointly strongly continuous family of bounded operators on X which satisfies all the usual algebraic properties of a propagator [11,44]. We stress that if the operators $A(\tau)$ in (0-1) are bounded on X, and if the function $\tau \mapsto \|A(\tau)\|$ is bounded, then the propagator $U(\tau, s)$ for (0-1) is defined for all $\tau, s \in \mathbb{R}$ and is an invertible operator [11]. Here, $\|\cdot\|$ denotes the operator norm on $B(X)$, the set of bounded operators on X. In general, however, the operators $U(\tau, s)$, $\tau \geq s$, are not invertible, and the operators $A(\tau)$ in (0-1) might be unbounded [44].

To any evolutionary family $\{U(\tau, s)\}_{\tau \geq s}$ on X, one can associate a so-called *evolutionary semigroup*, $\{e^{t\Gamma}\}_{t \geq 0}$, defined on $L_p(\mathbb{R}; X)$ $(p \geq 1)$ by the rule

$$(e^{t\Gamma} f)(\tau) = U(\tau, \tau - t) f(\tau - t), \quad t \geq 0, \quad \tau \in \mathbb{R}. \tag{0-3}$$

If, in particular, $\{U(\tau, s)\}_{\tau \geq s}$ is a smooth propagator [36] for (0-1), then the generator Γ of the evolutionary semigroup (0-3) is given by formula (0-2). This is true, in particular, if the function $\tau \mapsto \|A(\tau)\|$ is bounded [36].

Evolutionary semigroups have been the subject of intensive study (see [15,16,20, 30]), and significant progress on this topic has been made recently in [35,36] and [40,41].

This paper continues the investigation of evolutionary semigroups on $L_p(\mathbb{R}; X)$ as started in [26]. The related works [37,38,33,45] on dichotomy of differential equations, [32,39] on semigroup theory and [10,21,28], [18,42,43] on the theory of linear skew-product flows should also be mentioned.

For any evolutionary semigroup (0-3), the spectrum $\sigma(\Gamma)$ of its generator Γ is invariant under the translations along the imaginary axis and the spectral mapping theorem

$$\sigma(e^{t\Gamma}) \setminus \{0\} = e^{t\sigma(\Gamma)}, \quad t \geq 0 \tag{0-4}$$

is valid (see [26]). Hence, the generator Γ is invertible on $L_p(\mathbb{R}; X)$ if and only if the semigroup (0-3) is hyperbolic, that is $\sigma(e^{t\Gamma}) \cap \mathbb{T} = \emptyset$ for $t > 0$ and $\mathbb{T} = \{z : |z| = 1\}$.

In the present paper, with no restrictions on the Banach space, we prove the following hyperbolicity result. In the case $p = 2$ and X finite dimensional, this result appears in [6, Theorem 1.1], for $p = 2$ and X a Hilbert space, see [41, Theorem 8]; for X a separable Banach space, see [25,26].

THEOREM H. *The hyperbolicity of an evolutionary semigroup (0-3) (or, equivalently, the invertibility of* Γ*) is equivalent to the existence of an exponential dichotomy for the evolutionary family* $\{U(\tau, s)\}_{\tau \geq s}$.

The difficult step in the proof of this hyperbolicity result is proving that the Riesz projection $\mathcal{P}$ for the hyperbolic operator e^{Γ} corresponding to $\sigma(e^{\Gamma}) \cap \mathbb{D}$ ($\mathbb{D}$ being the unit disk) is an operator of multiplication by a strongly continuous bounded projection-valued function $P(\cdot)$ on $\mathbb{R}$:

$$(\mathcal{P}f)(\tau) = P(\tau)f(\tau), \quad \tau \in \mathbb{R}, \quad f \in L_p(\mathbb{R}; X). \tag{0-5}$$

This function $P(\cdot)$ defines the dichotomy projection for $\{U(\tau, s)\}_{\tau \geq s}$ (see Definition 3.2), and the result appears here as Lemma 3.5. In the case $p = 2$, we refer to [6, formula (1.17)] where X is finite dimensional, and to [41, Lemma 7] where X is a Hilbert space. See also [5, Theorem 2.1] for singular difference equations in finite-dimensional spaces; this paper also contains a discrete version of Theorem H for singular difference equations in finite-dimensional spaces. Analogous results for such equations also appear in [8].

With regard to the proof of (0-5), we note that e^{Γ} is a *weighted translation operator* and can be expressed as $e^{\Gamma} = aV$; the "weight" a is the operator of multiplication by the function $a(\tau) = U(\tau, \tau - 1)$, that is $(af)(\tau) = U(\tau, \tau - 1)f(\tau)$, and the translation operator V is defined as $(Vf)(\tau) = f(\tau - 1)$. The desired formula for $\mathcal{P}$ can be obtained from the Riesz integral formula as in [1,28,40,41] provided one knows that the resolvent operator, $b := (\lambda I - aV)^{-1}$, $\lambda \in \mathbb{T}$, belongs to the algebra of weighted translation operators. This means that b can be written as

$$b = \sum_{k=-\infty}^{\infty} a_k V^k, \quad \|b\|_1 := \sum_{k=-\infty}^{\infty} \|a_k\|_{B(L_p(\mathbb{R};X))} < \infty, \tag{0-6}$$

where $a_k \in \mathfrak{A}$, and $\mathfrak{A} := \mathfrak{A}(X)$ denotes the subalgebra in $B(L_p(\mathbb{R};X))$ of operators of multiplication by strongly continuous bounded operator-valued functions $a_k : \mathbb{R} \to B(X)$.

The previous paragraph provides motivation for studying the algebra of weighted translation operators. This is done in Section 2 where it is also shown that the algebra $\mathfrak{B}_*$ of operators (0-6) for which $a_k \in \mathfrak{A}(X)$ and $a_k^* \in \mathfrak{A}(X^*)$ is an inverse-closed subalgebra in $B(L_p(\mathbb{R};X))$. This means that if $b \in \mathfrak{B}_*$ and $b^{-1} \in B(L_p(\mathbb{R};X))$, then $b^{-1} \in \mathfrak{B}_*$. Moreover, we represent $\mathfrak{B}_*$ by weighted shift operators on $\ell_p(\mathbb{Z};X)$; that is, we associate with an operator $b \in \mathfrak{B}_*$ a family of operators $\pi_\tau(b)$, $\tau \in \mathbb{R}$, defined on $\ell_p(\mathbb{Z};X)$ by the formula

$$\pi_\tau(b) := \sum_{k=-\infty}^{\infty} \pi_\tau(a_k) S^k,$$

where

$$\pi_\tau(a)((x_n)_{n\in\mathbb{Z}}) = (a(\tau+n)x_n)_{n\in\mathbb{Z}}, \quad S((x_n)_{n\in\mathbb{Z}}) = (x_{n-1})_{n\in\mathbb{Z}}, \quad (x_n) \in \ell_p(\mathbb{Z};X).$$

We prove, that $b \in \mathfrak{B}_*$ is invertible if and only if $\pi_\tau(b)$ is invertible in $B(\ell_p(\mathbb{Z};X))$ for each $\tau \in \mathbb{R}$ and the function $\tau \mapsto \|[\pi_\tau(b)]^{-1}\|$ is bounded on $\mathbb{R}$.

We note that these two results may be of independent interest outside the theory of evolutionary semigroups (0-3). They are proven using ideas related to the important works [17,22,23] and make more precise the corresponding facts from [17,22,23] for the setting considered in the present paper. The technique of representing the algebra of weighted translation operators by weighted shift operators — or more concretely, the transition from differential equations to discrete systems — is a classical one and goes back to [2]; see also [1,13,22,29]. In connection with dichotomy, it also appears in [9, Section 5]

Having applied these results in Section 3 to the semigroup (0-3), we prove the hyperbolicity theorem, mentioned above. As an immediate consequence we give a one-line proof of the "roughness" of the exponential dichotomy.

In Section 4, Theorem H along with results from [4] are used to clarify, for *any* evolutionary family $\{U(\tau,s)\}_{\tau\geq s}$, some classical results that are known to hold in the case that the operators $A(\tau)$ in (0-1) are bounded (see [11]). We prove, for example, that the exponential dichotomy of $\{U(\tau,s)\}_{\tau\geq s}$ is equivalent to the existence and uniqueness of a