REGULARIZATION AND FREQUENCY–DOMAIN STABILITY OF WELL–POSED SYSTEMS

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Abstract. We study linear control systems with unbounded control and observation operators using certain regularization techniques. This allows us to introduce a modification of the transfer function for the system also if the input and output operators are not admissible in the usual sense. The modified transfer function is utilized to show exponential stability of sufficiently smooth solutions for the internal system under appropriate admissibility conditions on the system operators and appropriately modified stabilizability and detectability assumptions. If the internal system satisfies additional regularity properties, then we even obtain its uniform exponential stability.

1. Introduction

The topic of general infinite-dimensional linear systems has been studied by many authors focusing on a variety of classes and representations. Among the most general of these classes are the well-posed systems introduced by Salamon [18] and Weiss [23, 24, 25] which allow for unbounded control and observation operators (see, e.g., the survey [9]). A subclass of these well-posed systems is the set of regular linear systems. These were investigated by Weiss [23, 24, 25, 26], who showed that such systems allow nice generalizations of finite-dimensional systems by admitting the differential representation

$$x'(t) = Ax(t) + Bu(t), \quad y(t) = C_L x(t) \tag{1.1}$$

on Banach spaces X, U and Y, and the transfer function has a representation of the form $s \mapsto H(s) = C_L(s - A)^{-1}B$. Here, B is the control operator and C_L denotes the Lebesgue extension of the system observation operator, C; the extension is needed to account for the possibility that the domain of C may not contain $(s - A)^{-1}Bu$, $u \in U$.

This paper focuses on well-posed systems that are *not* necessarily regular, but a discussion of the latter is useful for putting our results in context. The main result concerns an

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equivalence between internal and external stability. This type of result has a history of predecessors and we refer to [7] and [28] for discussion and additional references. With respect to this result for regular systems, Rebarber [17] showed that internal (uniform exponential) stability is equivalent to stabilizability, detectability and external (input-output) stability. For general well-posed systems Morris [15] and Staffans [21] have formulated more general definitions of stabilizability and detectability and proved analogous results on internal versus external stability. These definitions and results lack the realization (1.1) of a regular system and its transfer function H; as such, they cannot be stated explicitly in terms of the transfer function given directly by the system operators. The most general theorem of this type for autonomous systems is by Weiss and Rebarber [28] which also avoids the assumption of regularity by replacing the concepts of stabilizability and detectability with the more general concepts of optimizability and estimatability. We refer the reader to that paper for a more detailed history of this result. For nonautonomous systems, results of this type were proven in [6] and, more generally, in [19], but here we address only autonomous systems.

The present goal is not to provide another generalization of these concepts but rather to retain, even for general well-posed systems, an explicit transfer-function-like criteria in terms of the operators A, B, C for external stability that can be used to infer an internal stability of the system. An additional consequence of our approach is that we allow for varying degrees of admissibility and unboundedness in the control and observation operators. To describe this, we briefly recall the concepts of generalized transfer functions and well-posed systems; details, references and extensions are given in Sections 3 and 4.

Let A generate a strongly continuous semigroup, $\{T(t)\}_{t\geq 0}$ on X, with domain $X_1 := D(A) \subset X$. We use the notation $R(\lambda, A) = (\lambda - A)^{-1}$ for λ in the resolvent set $\rho(A)$ of A, and let X_{-1} be the completion of X with respect to the norm $\|x\|_{-1} = \|R(\lambda, A)x\|$. If B and C are admissible control and observation operators, the generalized transfer functions of this triple (A, B, C) are solutions, $H : \rho(A) \to \mathcal{L}(U, Y)$, of the equation

$$\frac{H(\lambda) - H(\mu)}{\lambda - \mu} = -CR(\mu, A)R(\lambda, A)B, \tag{1.2}$$

$\lambda \neq \mu$, see, e.g., [9] and the discussion in Section 3 surrounding equation (3.10). Here $\mathcal{L}(U, Y)$ denotes the set of bounded linear operators from U into Y. Due to the admissibility of $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$, the right-hand side of (1.2) makes sense and any such H is an $\mathcal{L}(U, Y)$-valued function, analytic in some half-plane. If, in addition, the transfer functions are bounded in some right half-plane $\mathbb{C}_w = \{z \in \mathbb{C} : Re(z) > w\}$ then (A, B, C) is called well-posed. Unfortunately, this does not provide a nice transfer function realization in the classical form of $H(s) = C(s - A)^{-1}B$. As noted above, this drawback was overcome by Weiss [25] by focusing on the subclass of regular systems. On the other hand, while many systems arising in practice may indeed be regular, a proof of regularity poses additional complications and may require an explicit construction of C_L.

Rather than requiring the existence of a transfer function of the form H, we take our lead from the right-hand side of (1.2) and note that the operator $R(\mu, A)R(\lambda, A)$ is related to the ω_1-growth bound of the semigroup $T(t)$ in a similar way that the resolvent is related to the uniform exponential (ω_0) stability of the semigroup. That is, a semigroup on a Hilbert space is exponentially stable (i.e., the uniform growth bound $\omega_0(A)$ is negative) if and only if the resolvent, $R(\lambda, A)$, is bounded and analytic on the right half plane $\mathbb{C}_0$. Similarly, boundedness and analyticity of $\lambda \mapsto R(\mu, A)R(\lambda, A)$ on $\overline{\mathbb{C}_0}$ is equivalent to $\omega_1(A) < 0$; this determines the stability of solutions in the sense that if $\omega_1(A) < 0$, then all orbits starting in X_1 are exponentially stable (see Section 2). In view of this, the main theorem becomes:

for a well-posed system (A, B, C) on a Hilbert space, the condition $\omega_1(A) < 0$ is equivalent to the conditions of stabilizability, detectability and bounded analyticity of $\lambda \mapsto G_\mu(\lambda) = CR(\mu, A)R(\lambda, A)B$.

The above observations lead to a wide range of flexibility in formulating stability criteria by considering the entire scale of growth bounds, $\omega_\alpha(A)$, $\alpha \geq 0$ (see equation (2.1)). To exploit this we introduce in Section 4 varying degrees of admissibility for B and C that we call β-admissibility and γ-admissibility, respectively. The modified transfer function has the form $G_\mu(\lambda) := CR(\mu, A)^\gamma R(\lambda, A)R(\mu, A)^{\beta-1}B$, for a fixed $\mu \in \rho(A)$.

A scenario of interest concerns hyperbolic systems where the second time-derivatives are observed (e.g., acceleration in mechanical systems; cf. [3] and Example 5.5). One may consider $C \in \mathcal{L}(D(A^2), Y)$ to be a 2-admissible observation operator for A if $x \mapsto CT(\cdot)x$ extends to a bounded operator on $X_1 \to L^2([0, t], Y)$ $(t \geq 0)$. If, in addition, $B \in \mathcal{L}(U, X_{-1})$ is admissible in the usual sense (Section 3), we must expand the typical Gelfand triple $X_1 \hookrightarrow X \hookrightarrow X_{-1}$ (of interest provided both B and C are admissible in the usual sense) to $X_2 \hookrightarrow X \hookrightarrow X_{-1}$. In a setting such as this the main theorem becomes

the stability of solutions as defined by $\omega_2(A) < 0$ is equivalent to the conditions of stabilizability, detectability and bounded analyticity of $G_\mu(\lambda) = CR(\mu, A)^2 R(\lambda, A)B$.

We note that this equivalence involves the weaker concepts of $\omega_\alpha(A)$ and G_μ (versus $\omega_0(A)$ and $H(\lambda) = CR(\lambda, A)B$) allowing more general application of this type of result. In many settings (see Remark 2.1) we have $\omega_\alpha(A) = \omega_0(A)$, in which case the conclusion of internal stability is strictly stronger than traditional statements of this type: indeed, we obtain the conclusion $\omega_0(A) < 0$ under weaker hypotheses.

The paper is organized as follows. We first recall necessary prerequisites from the asymptotic theory of operator semigroups and from infinite dimensional control theory, having in mind readers familiar with only one of both fields. Section 2 establishes some notation and relevant background on spectral and growth bounds for semigroups and their generators. Section 3 provides some background and heuristics on admissible operators and well-posed systems (as developed by D. Salamon, G. Weiss and others) and proves a lemma needed for the generalizations of these concepts presented in Section 4. These generalizations are aimed at an explicit representation of a (modified) transfer function.

For this, we extend the concepts of admissibility by defining β–admissible control and γ–admissible observation operators ($\beta, \gamma \geq 1$). Here, $\beta = 1$ and $\gamma = 1$ correspond to the usual definitions of admissibility, and several examples illustrate natural settings in which $\beta > 1$ and $\gamma > 1$. This leads to the definition of a "(β, γ)–well–posed system" and the corresponding concepts of a (β, γ)–regularized transfer function and the modified transfer function (cf. (4.4) and (1.2)). The main theorem of the paper is proven in Section 5, Theorem 5.1. Of particular interest is a regularized version of internal stability (i.e., $\omega_{\beta+\gamma-1}(A) < 0$) under the condition that the system (A, B, C) is stabilizable and detectable. However, we also allow for regularized versions of the latter concepts, κ–stabilizability on X_b and ι–detectability on X_c, in order to deal with the increased irregularity of B and C, see Definition 4.9. As an illustration of these concepts, we treat the Laplacian on the d–dimensional torus with point observation and control in Example 4.12. The corresponding system operators B and C are admissible if and only if $\beta = \gamma > \max\{1, \frac{d}{4} + \frac{1}{2}\}$. For $d > 1$, the system is not well-posed in the traditional sense (i.e., not (1,1)-well-posed), but is (β, β)-well-posed for $\beta > \frac{d}{4} + \frac{1}{2}$. The modified and regularized transfer functions are computed, and it is checked that they are bounded on $\mathbb{C}_1$. The system is seen to be stabilizable in the sense of Definition 4.9. Moreover, for $d \in \{2, 3\}$ a (generalized) transfer function H exists, but it is unbounded in every right halfplane. The paper concludes with corollaries and discussion of the main theorem, and an example which uses the main theorem to prove the lack of detectability a weakly damped wave equation.

2. Semigroups and growth bounds

Let $T(\cdot)$ be a strongly continuous (linear) semigroup on a Banach space X generated by the operator A with domain $D(A)$. The *(uniform exponential) growth bound* of $T(\cdot)$ is defined by $\omega_0(A) = \inf\{a \in \mathbb{R} : \exists\, M = M(a) \geq 1 \text{ such that } \|T(t)\| \leq Me^{at},\ t \geq 0\}$. Thus $T(\cdot)$ is uniformly exponentially stable if and only if $\omega_0(A) < 0$. Fix some $w > \omega_0(A)$ and let $\alpha > 0$. Then we can define the bounded linear operator

$$(w - A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-wt} T(t) dt$$

on X. It can be proved that this map is injective; thus it has a closed inverse denoted by $(w - A)^{\alpha}$. Moreover, $(w - A)^{-\alpha}$ converges strongly to the identity I as $\alpha \to 0$, for $\alpha \in \mathbb{Z}$ we obtain the usual powers of $R(w, a)$ and $(w - A)$, and we have $(w - A)^{\beta}(w - A)^{\gamma}x = (w - A)^{\beta+\gamma}x$ for $\beta, \gamma \in \mathbb{R}$ and x belonging to the intersection of the domains of the three operators. For $\alpha \geq 0$, we introduce the Banach space $X_\alpha^A = X_\alpha = D((w - A)^{\alpha})$ with norm $\|x\|_\alpha^A = \|x\|_\alpha = \|(w - A)^{\alpha}x\|$. The superscript '$A$' will usually be suppressed unless it may lead to confusion. In particular, $X_1 = D(A)$ and $X_0 = X$. It is known that $X_\alpha \hookrightarrow X_\beta \hookrightarrow X$ for $\alpha \geq \beta \geq 0$ with continuous and dense embeddings. The operator $(w - A)^{\alpha}$ clearly commutes with A and $T(t)$ so that $T(\cdot)$ can be restricted to a C_0–semigroup $T_\alpha(\cdot)$

on X_α generated by the restriction $A_\alpha : X_{\alpha+1} \to X_\alpha$ of A. Moreover, the semigroups $T_\alpha(\cdot)$ and $T_\beta(\cdot)$ are similar via the isometric isomorphism $(w-A)^{\alpha-\beta} : X_\alpha \to X_\beta$.

We extend this scale of Banach spaces in the negative direction by introducing the new norm $\|x\|_{-1}^A = \|x\|_{-1} = \|R(w,A)x\|$ for $x \in X$. The completion $X_{-1}^A = X_{-1}$ of X with respect to this norm is called the *extrapolation space* for A. As usual we identify X with a dense subspace of X_{-1}. Since $T(t)$ and A commute with $R(w,A)$, we can extend $T(\cdot)$ to a strongly continuous semigroup $T_{-1}(\cdot)$ on X_{-1} generated by the unique continuous extension $A_{-1} : X \to X_{-1}$ of A. These operators are similar to $T(t)$ and A, respectively, via the isometric isomorphism $w - A_{-1} : X \to X_{-1}$. (In reflexive Banach spaces, one can describe X_{-1} equivalently via duality.)

Let $\alpha \in [0,1]$. We are now in the position to define the spaces

$$X_{\alpha-1} = X_{\alpha-1}^A = (X_{-1}^A)_\alpha^{A_{-1}} \quad \text{with} \quad X \hookrightarrow X_{\alpha-1} \hookrightarrow X_{-1}$$

and the restrictions $T_{\alpha-1}(t) : X_{\alpha-1} \to X_{\alpha-1}$ and $A_{\alpha-1} : X_\alpha \to X_{\alpha-1}$ of $T_{-1}(t)$ and A_{-1}, respectively. This C_0–semigroup and its generator coincide with the unique continuous extensions of $T(t)$ and A to $X_{\alpha-1}$, respectively. This procedure can be iterated (finitely many times). So we obtain Banach spaces X_α for $\alpha \le -N$ and every $N \in \mathbb{N}$ and C_0–semigroups $T_\alpha(\cdot)$ on X_α generated by $A_\alpha : X_{\alpha-1} \to X_\alpha$ which are extensions or restrictions of $T(\cdot)$ and A. Moreover, X_α is continuously and densely embedded in X_β provided $\alpha \ge \beta$, and the semigroups $T_\alpha(\cdot)$ and $T_\beta(\cdot)$ are similar via the isometric isomorphism $(w-A)^{\alpha-\beta} : X_\alpha \to X_\beta$. In particular, $\sigma(A_\alpha) = \sigma(A_\beta)$, where $\sigma(B) = \mathbb{C} \setminus \rho(B)$ is the spectrum of a linear operator B. Usually we will omit the subscript 'α' for the operators $T(t)$ and A. We remark that varying $w > \omega_0(A)$ yields the same spaces and operators, but gives equivalent norms. See [2] and [10] for detailed expositions of the above facts.

Now fix some $w > \omega_0(A)$. The *fractional (uniform exponential) growth bound* of the semigroup $T(\cdot)$ is defined by

$$\begin{aligned}\omega_\alpha(A) &= \inf\{a \in \mathbb{R} : \exists\, M = M(a) \ge 1 \;\text{ such that }\; \|T(t)(w-A)^{-\alpha}\| \le Me^{at},\ t \ge 0\} \\ &= \inf\{a \in \mathbb{R} : \exists\, M = M(a) \ge 1 \;\text{ such that }\; \|T(t)x\| \le Me^{at}\|x\|_\alpha,\ t \ge 0,\ x \in X_\alpha\}\end{aligned}$$

for $\alpha \ge 0$; see [16], [22]. In other words, if $\omega_\alpha(A) < 0$, then all orbits starting in X_α are exponentially stable. For instance, $\omega_1(A) < 0$ means that all orbits in $C^1(\mathbb{R}_+, X)$ are exponentially stable. Obviously,

$$\omega_\alpha(A) \le \omega_\beta(A) \le \omega_0(A) \tag{2.1}$$

if $\alpha \ge \beta \ge 0$, where strict inequality is possible, cf. [22, §4]. In order to describe the fractional growth bound in terms of (the resolvent of) A, we define the *spectral bound* and the *abscissa of growth order* α by

$$\begin{aligned} s(A) &= \sup\{\operatorname{Re}\lambda : \lambda \in \rho(A)\} \qquad \text{and} \\ s_\alpha(A) &= \inf\left\{a \ge s(A) : \sup_{\operatorname{Re}\lambda > a} \frac{\|R(\lambda,A)\|}{1 + |\operatorname{Im}\lambda|^\alpha} < \infty\right\} \end{aligned} \tag{2.2}$$

for $\alpha \geq 0$; see [16], [22]. Of particular interest is the abscissa of uniform boundedness $s_0(A)$. Clearly,

$$s(A) \leq s_\alpha(A) \leq s_\beta(A) \leq s_0(A) \tag{2.3}$$

for $\alpha \geq \beta \geq 0$. There are examples (even in Hilbert spaces) showing that $s(A) < s_0(A)$; see, e.g., [5, §2.1.5]. Due to [13, Lem.3.2], we have

$$s_\alpha(A) = \inf\{a \geq s(A) : \sup_{\operatorname{Re}\lambda > a} \|R(\lambda, A)(w - A)^{-\alpha}\| < \infty\}. \tag{2.4}$$

Since the resolvent of A is the Laplace transform of the semigroup, (2.4) implies that

$$s(A) \leq s_\alpha(A) \leq \omega_\alpha(A), \quad \alpha \geq 0, \tag{2.5}$$

for each generator A on a Banach space X. Again strict inequality may occur in this estimate as shown in Section 4 of [22]. On the other hand, Weis and Wrobel have established in Theorem 3.2 of [22] that

$$\omega_{\alpha+1}(A) \leq s_\alpha(A), \quad \alpha \geq 0. \tag{2.6}$$

In fact, they proved a more precise result. We say that a Banach space X has *Fourier type* $p \in [1, 2]$ if the Fourier transform is bounded from $L^p(\mathbb{R}, X)$ to $L^q(\mathbb{R}, X)$, where $1/p + 1/q = 1$. Obviously, every Banach space has Fourier type 1. A Banach space has Fourier type 2 if and only if it is isomorphic to a Hilbert space and L^r–spaces have Fourier type $\min\{r, s\}$ with $1/r + 1/s = 1$; see [16] and [22] for references for these results. Theorem 3.2 of [22] now says that

$$\omega_{\alpha-1+\frac{2}{p}}(A) \leq s_\alpha(A), \quad \alpha \geq 0, \tag{2.7}$$

if the underlying Banach space X has Fourier type $p \in [1, 2]$. Combined with (2.5), this inequality yields

$$\omega_\alpha(A) = s_\alpha(A), \;\; \alpha \geq 0, \quad \text{if } X \text{ is a Hilbert space.} \tag{2.8}$$

In particular, one obtains the well known and important equality $s_0(A) = \omega_0(A)$ for a C_0–semigroup on a Hilbert space (which is a consequence of Gearhart's theorem, [5], [16]).

We point out that all these quantities coincide if we know that $s(A) = \omega_0(A)$, because of (2.1), (2.3), and (2.5). This happens for several important classes of semigroups collected in the following remark.

Remark 2.1. If one of the following conditions holds, then $s(A) = s_\alpha(A) = \omega_\beta(A)$ for $\alpha, \beta \geq 0$.

(a) $t \mapsto T(t)$ is continuous in operator norm at some $t_0 > 0$, e.g., if $T(\cdot)$ is analytic or $T(t_0)$ is compact; see e.g. [10, Cor.IV.3.12].
(b) $T(\cdot)$ is essentially compact, i.e., $\|e^{-s(A)t}T(t) - K\| < 1$ for some $t > 0$ and a compact linear operator K; see e.g. [10, Thm.V.3.7].
(c) $T(\cdot)$ is a bounded group; see e.g. [10, Thm.IV.3.16].

(d) $X = L^p(\mu)$ or $X = C_0(\Omega)$ for some $p \in [1,\infty)$ and a σ–finite measure space (M,μ) or for a locally compact Haussdorff space Ω, and $T(t)f \geq 0$ for $t \geq 0$ and $f \geq 0$; see e.g. [16, Thm.3.5.3, 3.5.4].

If one looks at this list, one sees that many standard applications of semigroup theory lead to semigroups with $s(A) = \omega_0(A)$, but that nonconservative wave equations are not covered by the above conditions. In fact, Renardy exhibited a wave equation with a corresponding semigroup on a Hilbert space satisfying $s(A) < \omega_0(A)$; see, e.g., [5, Ex.2.26].

3. Well–posed systems and transfer functions

There are several more-or-less equivalent ways to introduce well–posed control systems as found, for example, in [8], [14], [15], [18], [21], [23], [24], [25], [26] (and [19] for nonautonomous systems). For us it is convenient to use admissible control and observation operators (rather than input and output maps) and to concentrate on frequency–domain concepts like the transfer function (rather than the input–output map).

As in the previous section, X, Y, U denote Banach spaces and A generates the C_0–semigroup $T(\cdot)$ on X. Thus there exists the associated scale of Banach spaces X_α and the extended, respectively restricted, semigroups and generators on X_α which we usually denote by the same symbols $T(t)$ and A. We fix numbers $M \geq 1$ and $w > \omega_0(A)$ such that $\|T(t)\| \leq Me^{wt}$ and use the same w to define the fractional powers of A.

Definition 3.1. *An operator $B \in \mathcal{L}(U, X_{-1})$ is an* admissible control operator *for A if for all $t > 0$ and $u \in L^2([0,t], U)$ the* input function

$$\Phi_t u := \int_0^t T(t-s)Bu(s)\,ds \tag{3.1}$$

takes values in X.

We note that the above integral is defined in X_{-1}. For an admissible B, it is easy to see that Φ_t is bounded from $L^2([0,t], U)$ to X with norm less than $M'e^{w't}$ for some $M' \geq M$ and $w' \geq w$ (where $w' = w$ if $w > 0$, $w' > 0$ is arbitrary if $w = 0$ and $w' = 0$ if $w < 0$). Moreover, $t \mapsto \Phi_t u$ is continuous in X. Observe that

$$\Phi_{t+s} u = T(t)\Phi_s u_1 + \Phi_t u_2 \tag{3.2}$$

for $t, s \geq 0$ if $u = u_1$ on $[0,s]$ and $u = u_2(\cdot - s)$ on $[s, s+t]$. If a family of bounded linear operators $\Phi_t : L^2([0,t], U) \to X$, $t \geq 0$, satisfies (3.2), then $(T(t), \Phi_t)_{t\geq 0}$ is called an *abstract control system*. Every abstract control system can be represented by a uniquely determined admissible control operator $B \in \mathcal{L}(U, X_{-1})$ as in Definition 3.1. See [18], [21], [24] for these facts.

Definition 3.2. *An operator $C \in \mathcal{L}(X_1, Y)$ is an* admissible observation operator *for A if for all $t \geq 0$ the* output function *$\Psi_t x := CT(\cdot)x$, defined on X_1, extends to a bounded operator $\Psi_t : X \to L^2([0,t], Y)$.*

Again one can verify that $\Psi_t x$ is continuous in Y if $x \in X_1$, that $\|\Psi_t\| \le M' e^{w't}$ (for a possibly larger M' and the same w'), and that

$$\Psi_{t+s}x = \Psi_s x \ \text{ on } [0,s] \quad \text{and} \quad \Psi_{t+s}x = [\Psi_t T(s)x](\cdot - s) \ \text{ on } [s, s+t] \tag{3.3}$$

for $t, s \ge 0$ and $x \in X$ if C is admissible. If a family of bounded linear operators $\Psi_t : X \to L^2([0,t], Y)$, $t \ge 0$, satisfies (3.3), then $(T(t), \Psi_t)_{t\ge 0}$ is called an *abstract observation system*. Such a system can be represented as in Definition 3.2 by a uniquely determined admissible observation operator $C \in \mathcal{L}(X_1, Y)$. In addition, the *Lebesgue extension* C_L of C is defined as the limit

$$C_L x := \lim_{t\to 0} \frac{1}{t} \int_0^t (\Psi_t x)(s)\, ds \tag{3.4}$$

for $x \in D(C_L) := \{x \in X : \text{the above limit exists in } Y\}$. In general, C_L is not closable in X. It is not difficult to prove that $C \subseteq C_L$ and that $T(s)x \in D(C_L)$ and $\Psi_t x(s) = C_L T(s)x$ for all $x \in X$ and all Lebesgue points $s \in [0,t]$ of the output function. See [18], [21], [23] for these facts.

The following lemma is known for $\alpha = 0$, [7, Lem.2.5]. It plays an important role in the proof of our main results. Recall that $\omega_\alpha(A) \ge s(A)$ by (2.5).

Lemma 3.3. *Let A be the generator of the C_0–semigroup $T(\cdot)$. Assume that $a > \omega_\alpha(A)$ for some $\alpha \ge 0$.*

(a) *If B is an admissible control operator, then the $\mathcal{L}(U,X)$–valued functions $\lambda \mapsto (w-A)^{-\alpha} R(\lambda, A)B$ and $\lambda \mapsto (1 + |\operatorname{Im}\lambda|^\alpha)^{-1} R(\lambda, A)B$ are bounded on $\mathbb{C}_a$.*

(b) *If C is an admissible observation operator, then the $\mathcal{L}(X,Y)$–valued functions $\lambda \mapsto CR(\lambda, A)(w-A)^{-\alpha}$ and $\lambda \mapsto (1 + |\operatorname{Im}\lambda|^\alpha)^{-1} CR(\lambda, A)$ are bounded on $\mathbb{C}_a$.*

Proof. It is well known that

$$R(\lambda, A)(w-A)^{-\alpha} = \int_0^\infty e^{-\lambda t} T(t)(w-A)^{-\alpha}\, dt$$

for $\operatorname{Re}\lambda > \omega_0(A)$. By analytic continuation, this equality in fact holds for $\operatorname{Re}\lambda \ge a > \omega_\alpha(A)$. We first show the boundedness of the first-mentioned functions in (a) and (b), respectively. This part of assertion (a) follows from the assumptions and the identities

$$\begin{aligned}(w-A)^{-\alpha} R(\lambda, A)Bz &= \int_0^\infty e^{-\lambda t} (w-A)^{-\alpha} T(t)Bz\, dt \\ &= \sum_{n=0}^\infty e^{-\lambda n} T(n)(w-A)^{-\alpha} \int_0^1 T(1-s)Be^{-\lambda(1-s)} z\, ds\end{aligned}$$

for $z \in U$ and $\operatorname{Re}\lambda > a$. Similarly,

$$CR(\lambda, A)(w-A)^{-\alpha} x = \sum_{n=0}^\infty \int_0^1 e^{-\lambda s} CT(s)[e^{-\lambda n} T(n)(w-A)^{-\alpha} x]\, ds$$

for $x \in D(A)$ implies the first part of (b). These arguments show in particular that $R(\lambda, A)B$ and $CR(\lambda, A)$ are bounded for $\operatorname{Re}\lambda \ge \omega_0(A) + 1$, so that the functions $(1+$

$|\operatorname{Im}\lambda|^\alpha)^{-1}R(\lambda,A)B$ and $(1+|\operatorname{Im}\lambda|^\alpha)^{-1}CR(\lambda,A)$ are bounded for $\operatorname{Re}\lambda \geq \omega_0(A)+1$. Finally, on the strip $a \leq \operatorname{Re}\lambda \leq \omega_0(A)+1$ the boundedness of the two functions in (a) and (b), respectively, is in fact equivalent. This can be seen as in Lemma 3.2 of [13], where the case $B = C = I$ was studied. (Here one has to use the boundedness of $R(\lambda,A)B$ and $CR(\lambda,A)$ for $\operatorname{Re}\lambda \geq \omega_0(A)+1$.) □

Let B and C be admissible control and observation operators for A, respectively. In terms of semigroup theory, $x(t) = \Phi_t u$ is the *mild solution* of the evolution equation

$$x'(t) = Ax(t) + Bu(t), \quad t \geq 0, \qquad x(0) = 0, \tag{3.5}$$

with input $u \in L^2_{loc}(\mathbb{R}_+, U)$, where the sum is defined in X_{-1}. Thus $y(t) = Cx(t) = C\Phi_t u$ should be the output of (3.5) under the observation operator C. However, in general, $x(t)$ does not belong to X_1 (or to $D(C_L)$) so that we cannot apply C (or C_L) directly. To circumvent this problem, we follow [18] and restrict ourselves to inputs $u \in C^1(\mathbb{R}_+, U)$ with $u(0) = 0$. It is easy to see that $x(\cdot)$ is differentiable in X with derivative

$$x'(t) = \int_0^t T(t-s)Bu'(s)\,ds \tag{3.6}$$

and $x(\cdot)$ satisfies (3.5), see also [18, Lem.2.5]. Equation (3.5) further implies that

$$x(t) = R(w,A)(wx(t) - x'(t)) + R(w,A)Bu(t).$$

We thus introduce the output $y(\cdot)$ of (3.5) by setting

$$y(t) := CR(w,A)(wx(t) - x'(t)) + H_w u(t) \tag{3.7}$$

$$= C\Big[\int_0^t T(t-s)Bu(s)\,ds - R(w,A)Bu(t)\Big] + H_w u(t), \quad t \geq 0, \tag{3.8}$$

for $u \in C^1(\mathbb{R}_+, U)$ with $u(0) = 0$, where the operator $H_w \in \mathcal{L}(U,Y)$ is not yet determined. In (3.8) we have employed (3.5); observe that the term in brackets belongs to $D(A)$. Using the function y given by (3.7), we further define the *input–output operator* $\mathbb{F} : u \mapsto y$ for the above class of inputs u. Taking Laplace transforms, we deduce from (3.7), (3.1), and (3.6) that

$$\hat{y}(\lambda) = CR(w,A)\,[wR(\lambda,A)B\hat{u}(\lambda) - R(\lambda,A)B\lambda\hat{u}(\lambda)] + H_w\hat{u}(\lambda) \tag{3.9}$$

for $\operatorname{Re}\lambda > \omega_0(A)$. Observe that $\mathbb{F}$ commutes with right translations (i.e., the operators $(S(t)f)(s) = f(s-t)$ for $s \geq t$ and $(S(t)f)(s) = 0$ for $s < t$). Thus the restrictions $\mathbb{F}_t$ of $\mathbb{F}$ to the time interval $[0,t]$ are well defined for $t \geq 0$. If these operators can be extended to bounded operators from $L^2([0,t],U)$ to $L^2([0,t],Y)$ (denoted by the same symbol), then there are uniformly bounded operators $H(\lambda) \in \mathcal{L}(U,Y)$ such that $\hat{y}(\lambda) = H(\lambda)\hat{u}(\lambda)$ holds for $\operatorname{Re}\lambda \geq a > \omega_0(A)$, see [3], [26]. In the above calculations we can replace w by μ with $\operatorname{Re}\mu > \omega_0(A)$. Setting $H_\mu = H(\mu)$ in (3.9), we arrive at the equation

$$H(\lambda) - H(\mu) = (\mu - \lambda)\,CR(\mu,A)R(\lambda,A)B \tag{3.10}$$

for $\operatorname{Re}\lambda, \operatorname{Re}\mu > \omega_0(A)$. These arguments motivate the concept of a "well–posed" system (see [8], [14], [18], [20], [21], [26]):

Definition 3.4. *Let X, Y, U be Banach spaces. Assume that A generates the semigroup $T(\cdot)$ on X, that B is an admissible control operator for A, and that C is an admissible observation operator for A.* A transfer function *of the system (A, B, C) is a function $H : \mathbb{C}_a \to \mathcal{L}(U, Y)$ satisfying* (3.10) *for some $a > \omega_0(A)$. The system is* well–posed *if it has a bounded transfer function on $\mathbb{C}_a$. We then denote the system by (A, B, C, H).*

Observe that two transfer functions for (A, B, C) only differ by a fixed operator and that they are analytic due to (3.10). We point out that, if U and Y are Hilbert spaces, then the well–posedness of (A, B, C) implies the boundedness of $\mathbb{F}_t$; see [8], [26].

Weiss has developed a different approach to input–output operators which we now want to explain. Define $\mathbb{F}_t : u \mapsto y$ as above by (3.8). Then one obtains

$$\mathbb{F}_{s+t}u = \mathbb{F}_s u_1 \;\text{ on } [0, s] \quad\text{and}\quad \mathbb{F}_{s+t}u = \Psi_t\Phi_s u_1(\cdot - s) + \mathbb{F}_t u_2(\cdot - s) \;\text{ on } [s, s+t] \tag{3.11}$$

for $t, s \geq 0$ and inputs $u \in C^1(\mathbb{R}_+, U)$ with $u = u_1$ on $[0, s]$, $u = u_2(\cdot - s)$ on $[s, s+t]$, and $u_1(0) = u_2(0) = 0$. If $\mathbb{F}_t$ can be extended to a bounded linear operator $\mathbb{F}_t : L^2([0, t], U) \to L^2([0, t], Y)$ for each $t \geq 0$, then equation (3.11) holds for every $u \in L^2([0, s+t], U)$. This means that $\mathbb{F}_t$, $t \geq 0$, are input–output maps in the sense of [25], [26]. We note that then $\|\mathbb{F}_t\| \leq M'e^{w't}$ (with the same w' as above but possibly a larger M'). Moreover, $(T(t), \Phi_t, \Psi_t, \mathbb{F}_t)_{t\geq 0}$ is an *abstract linear system* as defined in [25], [26].

Such a system is called *regular* (with feedthrough $D = 0$) if $\lim_{t\to 0} \frac{1}{t}\mathbb{F}_t u_z = 0$ for the constant input $u_z(s) = z$, $s \geq 0$, $z \in U$. This is the case if and only if $R(\lambda, A)Bz \in D(C_L)$ for all $z \in U$ and some/all $\lambda \in \rho(A)$. Then a transfer function and the input–output operator of (A, B, C) are given by

$$H(\lambda) = C_L R(\lambda, A)B, \;\; \operatorname{Re}\lambda > \omega_0(A), \qquad \text{and} \qquad \mathbb{F}_t u(\tau) = C_L\Phi_\tau u, \;\; \tau \in [0, t]; \tag{3.12}$$

see [25], [26]. It should be noted that the operator C_L is usually not explicitly given and that its definition (3.4) depends both on C and $T(t)$. Thus there may be no concrete representation of H even for regular systems.

4. Generalizations of well-posedness

In order to allow for less regular control and observation operators and to replace the transfer function by an object given in terms of A, B, C, the concepts discussed in the previous section will now be extended. We use the notation introduced in Section 2, and U and Y are Banach spaces.

Definition 4.1. *(a) Let $\beta \geq 1$. An operator $B \in \mathcal{L}(U, X_{-\beta})$ is called β–*admissible control operator *for A if the input function $\Phi_t u := \int_0^t T(t-s)Bu(s)\,ds$ takes values in $X_{1-\beta}$ for $t \geq 0$ and $u \in L^2_{loc}(\mathbb{R}_+, U)$.*
*(b) Let $\gamma \geq 1$. An operator $C \in \mathcal{L}(X_\gamma, Y)$ is a γ–*admissible observation operator *for A if for all $t \geq 0$ the* output function *$\Psi_t x := CT(\cdot)x$, defined on X_γ, extends to a bounded operator $\Psi_t : X_{\gamma-1} \to L^2([0, t], Y)$.*

Observe that 1–admissibility is just admissibility in the sense of Definitions 3.1 and 3.2. The theory presented in the previous two sections easily implies the following characterizations of generalized admissibility.

Proposition 4.2. *Let X, Y, U be Banach spaces and let $\beta, \gamma \geq 1$. Assume that A generates the C_0–semigroup $T(\cdot)$ on X with associated spaces $X_\alpha = X_\alpha^A$. Fix $w > \omega_0(A)$.*
(1) For $B \in \mathcal{L}(U, X_{-\beta})$ let Φ_t be given as in Definition 4.1. Then the following assertions are equivalent.

(a) *B is a β–admissible control operator for A.*
(b) *$\tilde{B} := (w - A_{-\beta})^{1-\beta} B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for A.*
(c) *$\tilde{\Phi}_t := (w - A_{-\beta})^{1-\beta} \Phi_t$, $t \geq 0$, is an abstract control system for A on X.*
(d) *B is an admissible control operator for $A_{1-\beta}$ on $X_{1-\beta}$.*
(e) *Φ_t, $t \geq 0$, is an abstract control system for $A_{1-\beta}$ on $X_{1-\beta}$.*

Conversely, if $\tilde{B} \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for A, then the operator $(w - A_{-\beta})^{\beta-1} \tilde{B} \in \mathcal{L}(U, X_{-\beta})$ is a β–admissible control operator for A.
(2) For $C \in \mathcal{L}(X_\gamma, Y)$ let Ψ_t be given as in Definition 4.1. Then the following assertions are equivalent.

(a) *C is a γ–admissible observation operator for A (with Lebesgue extension C_L).*
(b) *$\tilde{C} := C(w - A_1)^{1-\gamma} \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for A (with Lebesgue extension $\tilde{C}_L = C_L(w - A_1)^{1-\gamma}$).*
(c) *$\tilde{\Psi}_t := \Psi_t (w - A_1)^{1-\gamma}$, $t \geq 0$, is an abstract observation system for A on X.*
(d) *C is an admissible observation operator for $A_{\gamma-1}$ on $X_{\gamma-1}$.*
(e) *Ψ_t, $t \geq 0$, is an abstract observation system for $A_{\gamma-1}$ on $X_{\gamma-1}$.*

Conversely, if $\tilde{C} \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for A (with Lebesgue extension $\tilde{C}_L$), then $\tilde{C}(w - A_\gamma)^{\gamma-1} \in \mathcal{L}(X_\gamma, Y)$ is a γ–admissible observation operator for A (with Lebesgue extension $\tilde{C}_L(w - A_\gamma)^{\gamma-1}$).

One can interpret the above results in two ways. If B and C are β– and γ–admissible control and observation operators, respectively, then the corresponding abstract control and observation systems act on the larger state space $X_{1-\beta}$, respectively on the smaller state space $X_{\gamma-1}$. This looks somewhat awkward, but the picture becomes more agreeable if we look at the regularized operators $\tilde{B}$ and $\tilde{C}$ whose control and observation systems act on X itself. The following examples show that β– and γ–admissible control and observation operators arise quite naturally in standard situations.

Example 4.3. Consider the heat equation with Dirichlet boundary control and Neumann boundary observation:

$$\begin{aligned} \partial_t w(t,x) &= \Delta w(t,x), \qquad x \in \Omega,\ t \geq 0, \\ w(t,x) &= u(t,x), \qquad x \in \partial\Omega,\ t \geq 0, \\ y(t,x) &= \partial_\nu w(t,x), \qquad x \in \partial\Omega,\ t \geq 0, \\ w(0,x) &= w_0(x), \qquad x \in \Omega, \end{aligned} \tag{4.1}$$

on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, $\partial\Omega$. Let $X = L^2(\Omega)$, $w_0 \in H^2(\Omega)$, $Y = U = L^2(\partial\Omega)$, $u \in L^2_{loc}(\mathbb{R}_+, Y)$, and $Cf = \partial_\nu f \in Y$ be the trace on $\partial\Omega$ of the outer normal derivative of f. Let A be the Dirichlet Laplacian, i.e., $Af = \Delta f$, with $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. Since A generates an analytic semigroup and $C : X_{\varepsilon+3/4} \to Y$ is bounded for every $\varepsilon > 0$, see e.g. [11, §3.1], it follows that

$$\|C(-A)^{1-\gamma}T(t)\|^2 = \|C(-A)^{-\varepsilon-3/4}(-A)^{-\gamma+\varepsilon+7/4}T(t)\|^2 \le c\, t^{2\gamma-2\varepsilon-7/2}$$

is integrable near 0 for every $\gamma > 5/4$. This means that C is γ–admissible for $\gamma > 5/4$ by Proposition 4.2. We have to reformulate the boundary control employing the Dirichlet map $D : Y \to H^{1/2}(\Omega) \hookrightarrow X_{-\varepsilon+1/4}$ ($\varepsilon > 0$) defined by $v = D\varphi$ if $\Delta v = 0$ on Ω and $v = \varphi$ on $\partial\Omega$ (in the sense of distributions and trace); see, e.g., [11, (3.1.7)]. Setting $B = -A_{-1}D$, one can verify that a function $w \in L^2([0,T], H^2(\Omega)) \cap H^1([0,T], L^2(\Omega))$ solves (4.1) if and only if it satisfies

$$\begin{aligned} w'(t) &= A_{-1}w(t) + Bu(t), \qquad t \ge 0, \\ y(t) &= Cw(t), \qquad t \ge 0, \\ w(0) &= w_0 \end{aligned}$$

(cf. [11, §3.1] and [18]). Since $(-A_{-1})^{-\varepsilon-3/4}B : Y \to X$ is bounded by the properties of D and A, we see as above that B is β–admissible for every $\beta > 5/4$. $\diamond$

Example 4.4. We consider the wave equation with Neumann boundary control of the position and Dirichlet boundary observation of the velocity

$$\begin{aligned} \partial_{tt}\, w(t,x) &= \Delta w(t,x) - w(t,x), \qquad x \in \Omega,\ t \ge 0, \\ \partial_\nu w(t,x) &= u(t,x), \qquad x \in \partial\Omega,\ t \ge 0, \\ y(t,x) &= \partial_t w(t,x), \qquad x \in \partial\Omega,\ t \ge 0, \\ w(0,x) &= w_0(x),\ \ \partial_t\, w(0,x) = w_1(x), \qquad x \in \Omega, \end{aligned} \tag{4.2}$$

on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. Let $w_0 \in H^2(\Omega)$, $w_1 \in H^1(\Omega)$, $Y = U = L^2(\partial\Omega)$, $u \in L^2_{loc}(\mathbb{R}_+, Y)$. In order to put this problem in our framework, we use the space $X = H^1(\Omega) \times L^2(\Omega)$ and the Neumann Laplacian $\Delta_N f$ on $L^2(\Omega)$ with domain $D(\Delta_N) = \{f \in H^2(\Omega) : \partial_\nu f = 0 \text{ on } \partial\Omega\}$. We introduce $C(f,g)^T = g|\partial\Omega$, i.e., the trace operator acting on the second component and $A(f,g)^T = (g, (\Delta_N - I)f)^T$ with $D(A) = D(\Delta_N) \times H^1(\Omega)$. It is well known that A generates an unitary C_0–semigroup on X. It can be seen that $X_{-1} = L^2(\Omega) \times H^1(\Omega)^*$, where $H^1(\Omega)^*$ is the dual of $H^1(\Omega)$ with respect to the pivot space $L^2(\Omega)$. Moreover, $A_{-1}(f,g)^T = (g, (\Delta_N - I)f)^T$ for $(f,g)^T \in X$, where we write Δ_N instead of $(\Delta_N)_{-1/2}$. The operator $CA^{-1} : X \to Y$ is bounded since it is the trace operator acting on the first component. Hence, C is 2–admissible by Proposition 4.2. Further, let $N : Y \to H^{3/2}(\Omega)$ be the solution map of the elliptic boundary value problem $\Delta f - f = 0$ on Ω and $\partial_\nu f = \varphi$ on $\partial\Omega$ (see e.g. [11, (3.3.1.8)]). As in the previous example, we set $Bu = -(0, (\Delta_N - I)Nu)^T = -A_{-1}D$, where $Du = (Nu, 0)^T$ (see also [11, §8.6.1] or [18]). Since $A^{-1}B : Y \to X$ is bounded, B is 2-admissible by Proposition 4.2. Here the exponent 2 can be improved to $\beta > 7/5$ using deeper regularity results for this hyperbolic partial differential equation (see Lemma 8.6.1.1 of [11]). Due to Lemma 3.3.1.1 of [11] we have $B^* = -C$, so that C is γ–admissible for $\gamma > 7/5$ by duality. ◇

In accordance with (3.10), we define a *(β,γ)–regularized transfer function* $\tilde{H} : \mathbb{C}_a \to \mathcal{L}(U,Y)$ for (A,B,C) as a solution of the equation

$$\begin{aligned}\tilde{H}(\lambda) - \tilde{H}(\mu) &= (\mu-\lambda)\,\tilde{C}R(\mu,A)R(\lambda,A)\tilde{B}\\ &= (\mu-\lambda)\,C(w-A)^{2-\beta-\gamma}R(\mu,A)R(\lambda,A)B \qquad (4.3)\end{aligned}$$

for some $a > \omega_0(A)$. Here $w > \omega_0(A)$ is fixed. Note that $\tilde{H}$ is analytic by (4.3) and that two different (β,γ)–regularized transfer functions differ by a fixed operator.

Definition 4.5. *Let $\beta,\gamma \geq 1$. A (β,γ)–*well–posed system *$(A,B,C,\tilde{H})$ consists of a generator A on X, a β–admissible control operator B for A, a γ–admissible observation operator C for A, and a bounded (β,γ)–regularized transfer function $\tilde{H} : \mathbb{C}_a \to \mathcal{L}(U,Y)$ for (A,B,C) and some $a > \omega_0(A)$.*

Clearly, $(1,1)$–well–posedness is just well–posedness in the sense of Definition 3.4, where we may take $H = \tilde{H}$. It is easy to see that the results stated in the two previous sections allow to characterize the above concepts in terms of a regularized system.

Proposition 4.6. *Let X,Y,U be Banach spaces and let $\beta,\gamma \geq 1$. Assume that A generates the C_0–semigroup $T(\cdot)$ on X with associated spaces $X_\alpha = X_\alpha^A$, that B is a β–admissible control operator for A, and that C is a γ–admissible observation operator for A. For a fixed $w > \omega_0(A)$ we define $\tilde{B} := (w - A_{-\beta})^{1-\beta}B$ and $\tilde{C} := C(w - A_1)^{1-\gamma}$. Then (A,B,C) is (β,γ)–well–posed if and only if $(A,\tilde{B},\tilde{C})$ is well–posed. A (β,γ)–regularized transfer function for (A,B,C) is a transfer function for $(A,\tilde{B},\tilde{C})$, and vice versa.*

In our main result, instead of $\tilde{H}$, we use the function

$$\begin{aligned}G_w(\lambda) = G(\lambda) &:= C(w-A)^{1-\beta-\gamma}R(\lambda,A)B = \tilde{C}R(w,A)R(\lambda,A)\tilde{B},\ \operatorname{Re}\lambda > \omega_0(A), \quad (4.4)\\ &= \frac{\tilde{H}(\lambda)-\tilde{H}(w)}{w-\lambda}, \quad \lambda \neq w,\ \operatorname{Re}\lambda > \omega_0(A). \qquad (4.5)\end{aligned}$$

Definition 4.7. *Let X,Y,U be Banach spaces, $\beta,\gamma \geq 1$, A be a generator on X, B be a β–admissible control operator for A, and C be a γ–admissible observation operator for A. Fix $w > \omega_0(A)$. The function G defined in (4.4) is called the* modified transfer function *for (A,B,C).*

Observe that G is given quite explicitly in terms of the operators A,B,C, in contrast to $\tilde{H}$ (or H). Its definition does not require the existence of a (regularized) transfer function for (A,B,C). In the setting of Example 4.3 we obtain

$$G(\lambda) = C(-A)^{1-\beta-\gamma}R(\lambda,A)B = \partial_\nu(-A)^{2-\beta-\gamma}R(\lambda,A)D;$$

see also Example 4.12 below. In our main results we will assume, in particular, the boundedness of G on $\mathbb{C}_0$. This property corresponds to linear growth of $\tilde{H}$ as $\lambda \to \infty$ by

(4.5). Hence, it is a weaker statement than *external stability* of the regularized system, i.e., boundedness of $\tilde H$ on $\mathbb{C}_0$.

In view of the above proposition, there exists the input–output operator $\tilde{\mathbb{F}}$ (given by (3.7)) of the regularized system $(A,\tilde B,\tilde C,\tilde H)$ if we have a (β,γ)–well–posed system. We can thus define the *regularized output* of $(A,B,C,\tilde H)$ by $\tilde y = \tilde{\mathbb{F}}u$. Then $\hat{\tilde y}(\lambda) = \tilde H(\lambda)\hat u(\lambda)$ for $\operatorname{Re}\lambda > \omega_0(A)$, and formulas (3.7), (3.6), and (3.8) imply

$$\begin{aligned}\tilde y(t) &= C(w-A)^{1-\beta-\gamma}\int_0^t T(t-s)B[wu(s)-u'(s)]\,ds + \tilde H(w)u(t)\\ &= C(w-A)^{2-\beta-\gamma}\Big(\int_0^t T(t-s)Bu(s)\,ds - R(w,A)Bu(t)\Big) + \tilde H(w)u(t), \quad t\ge 0,\end{aligned}$$

for $u\in C^1(\mathbb{R}_+,U)$ with $u(0)=0$. If $(A,\tilde B,\tilde C,\tilde H)$ is regular, we obtain the representations

$$\tilde H(\lambda) = C_L(w-A)^{2-\beta-\gamma}R(\lambda,A)B \quad\text{and}\quad \tilde{\mathbb{F}}u(t) = C_L(w-A)^{2-\beta-\gamma}\int_0^t T(t-s)Bu(s)\,ds$$

due to (3.12) and Proposition 4.2. For our main result we need the following estimate for a 'modified regularized input–output operator'.

Lemma 4.8. *Let X,Y,U be Banach spaces, $\beta,\gamma\ge 1$, A be a generator on X, B be a β–admissible control operator for A, and C be a γ–admissible observation operator for A. Assume that $\omega_{\beta+\gamma-1}(A)<0$. For $u\in L^2(\mathbb{R}_+,U)$ we define the function*

$$z(t) = C(w-A)^{1-\gamma}R(w,A)\int_0^t T(t-s)(w-A)^{1-\beta}Bu(s)\,ds, \quad t\ge 0.$$

Then $\|z\|_{L^2(\mathbb{R}_+,Y)} \le c\,\|u\|_{L^2(\mathbb{R}_+,U)}$ for a constant $c>0$.

Proof. Using the operators $\tilde B = (w-A)^{1-\beta}B$ and $\tilde C = C(w-A)^{1-\gamma}$, we can restrict attention to the case that $\beta=\gamma=1$, i.e., B and C are admissible and $\omega_1(A) < -\delta < 0$. Let $t\in[n,n+1)$ for some $n\in\mathbb{N}$. Then

$$\begin{aligned}z(t) &= CR(w,A)\int_n^t T(t-s)Bu(s)\,ds\\ &\quad + \sum_{k=1}^n CT(t-n)T(n-k)R(w,A)\int_{k-1}^k T(k-s)Bu(s)\,ds,\\ \|z\|_{L^2([n,n+1],Y)} &\le c\,\|u\|_{L^2([n,n+1],U)} + c\sum_{k=1}^n e^{-\delta(n-k)}\|u\|_{L^2([k-1,k],U)}\end{aligned}$$

for a constant $c>0$. Setting $a_k = e^{-\delta k}$ if $k\in\mathbb{N}_0$ and $a_k=0$ otherwise, and $b_k = \|u\|_{L^2([k,k+1],U)}$ if $k\in\mathbb{N}_0$ and $b_k=0$ otherwise, we obtain

$$\|z\|^2_{L^2(\mathbb{R}_+,Y)} \le c^2\,e^{2\delta}\|(a_k)*(b_k)\|^2_{\ell^2(\mathbb{Z})}\,.$$

So Young's inequality implies the assertion. $\square$

We need two more concepts for our main results.

Definition 4.9. *Let X, Y, U be Banach spaces, $\beta, \gamma \geq 1$, $1 - \beta \leq b \leq 0$, $0 \leq c \leq \gamma - 1$, $\iota, \kappa \geq 0$, A be a generator on X, B be a β–admissible control operator for A, and C be a γ–admissible observation operator for A.*

(a) *(A, B) is called κ–*stabilizable on X_b^A *if there exists a C_0–semigroup $T^K(\cdot)$ on X_b^A with generator A^K and an admissible observation operator $K \in \mathcal{L}(D(A^K), U)$ for A^K such that $\omega_\kappa(A^K) < 0$ on X_b^A and*

$$R(\lambda, A^K)x = R(\lambda, A)x + R(\lambda, A)BKR(\lambda, A^K)x \tag{4.6}$$

for $x \in X_b^A$, $\operatorname{Re}\lambda > a$, and some $a \in \mathbb{R}$, where the equality holds in $X_{1-\beta}^A$.

(b) *(A, C) is called ι–*detectable on X_c^A *if there exists a C_0–semigroup $T^J(\cdot)$ on X_c^A with generator A^J and an admissible control operator $J \in \mathcal{L}(Y, (X_c^A)_{-1}^{A^J})$ for A^J such that $\omega_\iota(A^J) < 0$ on X_c^A and*

$$R(\lambda, A^J)x = R(\lambda, A)x + R(\lambda, A^J)JCR(\lambda, A)x \tag{4.7}$$

for $x \in X_{\gamma-1}^A$, $\operatorname{Re}\lambda > a$, and some $a \in \mathbb{R}$, where the equality holds in X_c^A.

Remark 4.10. We are mostly interested in the case where $\iota = \kappa = 0$, but we need the more general definition given above to state the implication (b) in our main Theorem 5.1. For $\beta = \gamma = 1$ we have of course $b = c = 0$. It seems to be most natural to consider $b = 1 - \beta$ and $c = \gamma - 1$, that is, to look for stablizability and detectability in the spaces $X_{1-\beta}$ and $X_{\gamma-1}$, respectively. First, then the equations (4.6) and (4.7) are understood in the space from which x is taken, respectively. Second, in the setting of Definition 4.9(a), if $(A_{1-\beta}, B, K)$ is a regular system on $X_{1-\beta}$ with $U = Y$ and if I is an admissible feedback for this system, then there is a generator A^K on $X_{1-\beta}$ satisfying (4.6) such that K is an admissible observation operator for A^K on $X_{1-\beta}$, due to [21, Chap.7] or [27]. An analogous fact holds for Definition 4.9(b). Moreover, the conclusion of Theorem 5.1(a) is stronger if we choose $b = 1 - \beta$ and $c = \gamma - 1$. On the other hand, it could be easier to work on the given space X itself instead of the usually more complicated spaces $X_{1-\beta}$ and $X_{\gamma-1}$, so that we also want to treat the case $b = c = 0$. See also Examples 4.12 and 5.5.

The next result shows that the equations (4.6) and (4.7) can be formulated equivalently in the time domain.

Proposition 4.11. *Let X, Y, U be Banach spaces, $\beta, \gamma \geq 1$, $1 - \beta \leq b \leq 0$, $0 \leq c \leq \gamma - 1$, $\iota, \kappa \geq 0$, A be a generator on X, B be a β–admissible control operator for A, and C be a γ–admissible observation operator for A. Let $T^K(\cdot)$ and $T^J(\cdot)$ be C_0–semigroups on X_b^A and X_c^A generated by the operators A^K and A^J, respectively, let $K \in \mathcal{L}(D(A^K), U)$ be an admissible observation operator for A^K, and let $J \in \mathcal{L}(Y, (X_c^A)_{-1}^{A^J})$ be an admissible control operator for A^J. Then (4.6) holds if and only if*

$$T^K(t)x = T(t)x + \int_0^t T(t-s)BK_LT^K(s)x\,ds \tag{4.8}$$

for all $t \geq 0$ and $x \in X_b^A$, where the equation is understood in $X_{1-\beta}^A$. Similarly, (4.7) *holds if and only if*

$$T^J(t)x = T(t)x + \int_0^t T^J(t-s)JC_LT(s)x\,ds \tag{4.9}$$

for all $t \geq 0$ and $x \in X_{\gamma-1}^A$, where the equation is understood in X_c^A.

Proof. First observe that by approximation we can restrict ourselves in (4.8) to $x \in D(A^K)$ and in (4.9) to $x \in X_\gamma^A$. So we can use K and C and instead of their Lebesgue extensions. Assume that (4.6) holds. Then the Laplace transform in $X_b^A \hookrightarrow X_{-\beta}^A$ of the function $T^K(\cdot)x - T(\cdot)x$, $x \in D(A^K)$, is given by $R(\lambda, A^K)x - R(\lambda, A)x$ for $\operatorname{Re}\lambda > \omega_0(A)$. On the other hand, applying Fubini's theorem twice, we obtain

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\int_0^t T(t-s)BKT^K(s)x\,ds\,dt &= \int_0^\infty \int_s^\infty e^{-\lambda(t-s)}T(t-s)BKe^{-\lambda s}T^K(s)x\,dt\,ds \\ &= \int_0^\infty \int_0^\infty e^{-\lambda r}T(r)BKe^{-\lambda s}T^K(s)x\,ds\,dr \\ &= R(\lambda, A)BKR(\lambda, A^K)x.\end{aligned}$$

Here the Laplace integral is defined in $X_{-\beta}^A$. Therefore, (4.8) is valid due to the uniqueness of the Laplace transform. If (4.8) holds, then (4.6) follows directly by taking Laplace transforms in $X_{-\beta}$. The second assertion is shown similarly using the Laplace transform in $X_c^A \hookrightarrow (X_c^A)_{-1}^{A^J}$. □

As a result, 0–stabilizability on X and 0–detectability on X are just the autonomous versions of *stabilizability* and *detectability* as introduced in Definitions 5.7 and 5.8 of [19] for nonautonomous systems (if $\beta = \gamma = 1$). Moreover, optimizability and estimatability (as defined in [28]) imply 0–stabilizability on X and 0–detectability on X due to formulas (3.10) and (4.11) in [28]. One can find several variants of the concepts 'stabilizability' and 'detectability' in the literature (see e.g. [7], [15], [17], [21]) which are (mostly) stronger than ours since usually well–posedness or regularity of the respective closed–loop systems is required. The following example illustrates the notions introduced in this section.

Example 4.12. Consider a control system governed by the Laplace operator with periodic boundary conditions and point control and observation. Thus we look at the problem $x_t = \Delta x + \mathcal{B}u$, $y = \mathcal{B}^*x$, where $x = x(t, \boldsymbol{\xi})$, $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d) \in \mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ (the d-dimensional torus), $d \geq 1$, and $\mathcal{B}^*$ is the point evaluation at $\boldsymbol{\xi} = \mathbf{0} = (0, \ldots, 0)$. Let $H^\alpha(\mathbb{T}^d; \mathbb{C})$, $\alpha \in \mathbb{R}$, denote the Sobolev space which, via the Fourier transform

$$v(\boldsymbol{\xi}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} v_{\mathbf{k}} e^{i\mathbf{k}\cdot\boldsymbol{\xi}} \mapsto \mathbf{v} = (v_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}, \quad v_{\mathbf{k}} := (2\pi)^{-d}\int_{\mathbb{T}^d} e^{i\mathbf{k}\cdot\boldsymbol{\xi}} v(\boldsymbol{\xi})\,d\boldsymbol{\xi},$$

we will identify with the sequence space $\ell_\alpha^2 = \ell_\alpha^2(\mathbb{Z}^d; \mathbb{C})$, $\alpha \in \mathbb{R}$, defined as

$$\ell_\alpha^2 := \Big\{\mathbf{v} = (v_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d} : \|\mathbf{v}\|_{\ell_\alpha^2}^2 := \sum_{\mathbf{k}\in\mathbb{Z}^d} (1 + \|\mathbf{k}\|^2)^\alpha |v_{\mathbf{k}}|^2 < \infty\Big\},$$

$\|\mathbf{k}\|^2 = \sum_{j=1}^d k_j^2$, $\mathbf{k} = (k_1, \dots, k_d)$. Under this identification, the Laplacian becomes $A : (v_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d} \mapsto (-\|\mathbf{k}\|^2 v_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$, the observation operator $\mathcal{B}^*$, $\mathcal{B}^* v = v(\mathbf{0})$, is transformed into $C : (v_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d} \mapsto \sum_{\mathbf{k}\in\mathbb{Z}^d} v_{\mathbf{k}}$, and the control operator $\mathcal{B}$ becomes B, defined via duality by $\langle \mathbf{v}, Bz\rangle = z \sum_{\mathbf{k}\in\mathbb{Z}^d} v_{\mathbf{k}}$ for $z \in \mathbb{C}$. If sequences $\mathbf{v}$ are viewed as "columns" $(v_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$, then $C : \ell^2_\alpha \to \mathbb{C}$ is a "row" $(\dots 1, 1, \dots) = \mathbb{1}^T$, while the control operator $B : \mathbb{C} \to \ell^2_\alpha : z \mapsto (z)_{\mathbf{k}\in\mathbb{Z}}$ is the transposed "column" $\mathbb{1} = (\dots 1, 1, \dots)^T$. We thus consider the control system (A, B, C) on $X = \ell^2$ and $Y = U = \mathbb{C}$. Since $D((I - \Delta)^\alpha) = H^{2\alpha}$ we have $X_\alpha = \ell^2_{2\alpha}$ for $\alpha \in \mathbb{R}$.

We claim that B is a β-admissible, $\beta \geq 1$, control operator for A in the sense of Definition 4.1(a) if and only if $\beta > \frac{d}{4} + \frac{1}{2}$. Hence, if $d = 1$ then B is an admissible control operator for A in the sense of Definition 3.1, while if $d = 2$ then B is not 1-admissible, but 2-admissible. Similarly, we claim that C is a γ-admissible, $\gamma \geq 1$, observation operator for A in the sense of Definition 4.1(b) if and only if $\gamma > \frac{d}{4} + \frac{1}{2}$. Indeed, to verify that $B \in \mathcal{L}(U, X_{-\beta})$, we note that for $z \in \mathbb{C}$ the series

$$\|Bz\|^2_{\ell^2_{-2\beta}} = |z|^2 \sum_{\mathbf{k}\in\mathbb{Z}^d} (1 + \|\mathbf{k}\|^2)^{-2\beta}$$

converges if and only if $\beta > \frac{d}{4}$ (by passing to spherical coordinates in the corresponding d-dimensional improper integral). Thus, $C = B^* \in \mathcal{L}(X_\gamma, \mathbb{C})$ if and only if $\gamma > \frac{d}{4}$. To see whether $\Phi_t u \in X_{1-\beta}$ for $t \geq 0$ and $u \in L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathbb{C})$, we use the Cauchy-Schwarz inequality to compute:

$$\begin{aligned}
\|\Phi_t u\|^2_{\ell^2_{2(1-\beta)}} &= \sum_{\mathbf{k}\in\mathbb{Z}^d} (1 + \|\mathbf{k}\|^2)^{2(1-\beta)} \left| \int_0^t e^{-(t-s)\|\mathbf{k}\|^2} u(s) ds \right|^2 \\
&\leq \|u\|^2_{L^2} \sum_{\mathbf{k}\in\mathbb{Z}^d} (1 + \|\mathbf{k}\|^2)^{2(1-\beta)} \int_0^t e^{-2(t-s)\|\mathbf{k}\|^2} ds \\
&\leq \Big(t + \sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} (1 + \|\mathbf{k}\|^2) \Big)^{1-2\beta} \|u\|^2_{L^2}.
\end{aligned}$$

As above, the last sum converges provided $4\beta > 2 + d$. Using a constant control u, we conclude that this condition is equivalent to $\Phi_t u \in \ell^2_{2(1-\beta)}$. This proves the β–admissibility of B and C for $\beta > \frac{d}{4} + \frac{1}{2}$ and $\beta \geq 1$.

Now we look for transfer functions. If $\gamma = \beta > \frac{d}{4} + \frac{1}{2}$, then the sum

$$\tilde{H}(\lambda) = \sum_{\mathbf{k}\in\mathbb{Z}^d} (1 + \|\mathbf{k}\|^2)^{2-2\beta} (\lambda + \|\mathbf{k}\|^2)^{-1}$$

converges for $\operatorname{Re}\lambda > 0$ and is uniformly bounded on, say, $\mathbb{C}_1$. It is easy to check that $\tilde{H}$ is a (β, β)–regularized transfer function for (A, B, C). Therefore the system $(A, B, C, \tilde{H})$ is (β, β)–well–posed for $\beta > \frac{d}{4} + \frac{1}{2}$ and $\beta \geq 1$; in particular, it is well–posed in the sense of Definition 3.4 if $d = 1$. Observe that the natural candidate for a transfer function

$$CR(\lambda, A)B = \sum_{\mathbf{k}\in\mathbb{Z}^d} (\lambda + \|\mathbf{k}\|^2)^{-1}\,, \qquad \operatorname{Re}\lambda > 0,$$

yields a divergent sum for every λ if $d \geq 2$. However, for $d = 2, 3$ the function

$$H(\lambda) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \frac{1-\lambda}{(\lambda + \|\mathbf{k}\|^2)(1 + \|\mathbf{k}\|^2)}\,, \qquad \operatorname{Re}\lambda > 0,$$

is a transfer function for (A, B, C). But H is unbounded on every right halfplane because $H(n)$ behaves as $\log n$ if $d = 2$ and as $\sqrt{n}$ if $d = 3$ as $n \to \infty$. (Use again the corresponding integral to check this fact.)

Next, for $\beta = \gamma \geq 1$ and $\beta > \frac{d}{4} + \frac{1}{2}$, the modified transfer function $G(\lambda) = C(1 - A)^{1-2\beta}R(\lambda, A)B$ is given by

$$G(\lambda) = \sum_{\mathbf{k}\in\mathbb{Z}^d} (1 + \|\mathbf{k}\|^2)^{1-2\beta}(\lambda + \|\mathbf{k}\|^2)^{-1}.$$

If $\operatorname{Re}\lambda \geq 1$ then the last sum converges and gives an analytic function bounded in $\mathbb{C}_1$ (this is even true for $\beta > d/4$). Clearly, G is not uniformly bounded on $\mathbb{C}_0$ which corresponds to the fact that $\sigma(A) = \{-\|\mathbf{k}\|^2 : \mathbf{k} \in \mathbb{Z}^d\}$.

Finally, (A, B) is 0-stabilizable on $X_{1-\beta}$ in the sense of Definition 4.9(a) with the operator $K : (v)_{\mathbf{k}\in\mathbb{Z}^d} \mapsto -v_{\mathbf{0}}$. Indeed, consider on $X_{1-\beta} = \ell^2_{2(1-\beta)}$ the operator A^K, formally defined as $A^K : (v_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d} \mapsto (w_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ with $w_{\mathbf{0}} = -v_{\mathbf{0}}$ and $w_{\mathbf{k}} = -\|\mathbf{k}\|^2 v_{\mathbf{k}} - v_{\mathbf{0}}$ for $\mathbf{k} \neq \mathbf{0}$. Since $K \in \mathcal{L}(X_{1-\beta}, \mathbb{C})$ and B is β-admissible for A, we conclude that

$$\int_0^t T(t-s)BKf(s)ds \in X_{1-\beta} \quad \text{for } f \in L^2([0, t]; X_{1-\beta}) \tag{4.10}$$

and all $t \geq 0$. Therefore $A^K := (A_{-\beta} + BK) \mid_{X_{1-\beta}}$ generates a strongly continuous semigroup $T^K(\cdot)$ on $X_{1-\beta}$ satisfying (4.6) due to the Desch-Schappacher perturbation theorem (see, e.g., Corollary III.3.4 and Equation (III.3.8) in [10]). Observe that A generates an analytic semigroup on $X_{1-\beta}$ which implies that $T^K(\cdot)$ is also analytic (cf. [10, Exer.III.3.8]). Since $\sigma(A^K) = \{-\|\mathbf{k}\|^2 : \mathbf{k} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}\}$, the semigroup $T^K(\cdot)$ is uniformly exponentially stable on $X_{1-\beta}$. The above argument can be modified to deduce the stabilizability of (A, B) on X if $d \in \{1, 2, 3\}$. (One has to use $f \in L^q([0, t]; X_{1-\beta})$ in (4.10) for a sufficiently large $q < \infty$.) For $d \geq 4$, the operator A^K does not generate a semigroup on X since one even has $(\lambda - A^K)^{-1}(1, 0, \cdots)^T \notin \ell^2$. ◇

5. Internal and external stability

We now come to the main result of this paper, which we discuss after the proof. We recall that we are mainly interested in the cases $\iota = \kappa = 0$, $b = 1 - \beta \leq 0$ (or $b = 0$), and $c = \gamma - 1 \geq 0$ (or $c = 0$), cf. Remark 4.10. We abbreviate $X_b = X_b^A$ and $X_c = X_c^A$.

Theorem 5.1. *Let X, Y, U be Banach spaces, $\beta, \gamma \geq 1$, $1-\beta \leq b \leq 0$, $0 \leq c \leq \gamma - 1$, $\iota, \kappa \geq 0$, A be a generator on X, B be a β–admissible control operator for A, and C be a γ–admissible observation operator for A. Set $\alpha = \beta + b + \gamma - c + \kappa + \iota - 1$ and fix $w > \omega_0(A)$.*

(a) *Assume that (A, B) is κ–stabilizable on X_b, that (A, C) is ι–detectable on X_c, and that the modified transfer function $G(\lambda) = C(w-A)^{1-\beta-\gamma}R(\lambda, A)B \in \mathcal{L}(U, Y)$ has a bounded analytic continuation to $\mathbb{C}_{-\varepsilon}$ for some $\varepsilon > 0$. Then we have $s_\alpha(A) < 0$ and, hence, $\omega_{\alpha+1}(A) < 0$. Moreover, if X is a Hilbert space, then $\omega_\alpha(A) < 0$; if X has Fourier type $p \in [1, 2]$, then $\omega_{\alpha-1+2/p}(A) < 0$.*

(b) *Conversely, if $\omega_{\beta+\gamma-1}(A) < 0$, then G has a bounded analytic continuation to $\mathbb{C}_{-\varepsilon}$ for some $\varepsilon > 0$ and (A, B, C) is $(\beta+\gamma-1)$–stabilizable on X_b and $(\beta+\gamma-1)$–detectable on X_c (for every $b \in [1-\beta, 0]$ and $c \in [0, \gamma-1]$).*

Proof. (a) In view of (2.6), (2.7), and (2.8), it remains to show that $s_\alpha(A) < 0$. This is done in four steps.
(1.i) Since (A, C) is ι–detectable on X_c, there is a generator A^J on X_c and an admissible control operator $J \in \mathcal{L}(Y, (X_c)_{-1}^{A^J})$ for A^J such that $s(A^J) \leq \omega_\iota(A^J) < 0$ and

$$R(\lambda, A^J)(w-A)^{1-\beta-\gamma}Bz = R(\lambda, A)(w-A)^{1-\beta-\gamma}Bz + R(\lambda, A^J)JCR(\lambda, A)(w-A)^{1-\beta-\gamma}Bz \qquad (5.1)$$

for large $\operatorname{Re}\lambda$ and $z \in U$. Observe that $x = (w-A)^{1-\beta-\gamma}Bz \in X_{\gamma-1} \hookrightarrow X_c$, and the above equation holds in X_c. Using the inequality $s(A^J) < 0$ and the analytic continuation of G, we can extend the left hand side and the second summand on the right hand side of (5.1) to analytic functions on a halfplane $\mathbb{C}_{-\eta}$ for some $\eta > 0$. Therefore the function $\lambda \mapsto R(\lambda, A)(w-A)^{1-\beta-\gamma}B \in \mathcal{L}(U, X_c)$ possesses an analytic continuation to $\mathbb{C}_{-\eta}$.
(1.ii) The κ–stabilizability of (A, B) on X_b yields a generator A^K on X_b and an admissible observation operator $K \in \mathcal{L}(D(A^K), U)$ for A^K such that $s(A^K) \leq \omega_\kappa(A^K) < 0$ and

$$(w-A)^{1-\beta-\gamma}R(\lambda, A^K)x = (w-A)^{1-\beta-\gamma}R(\lambda, A)x + R(\lambda, A)(w-A)^{1-\beta-\gamma}BKR(\lambda, A^K)x \qquad (5.2)$$

for large $\operatorname{Re}\lambda$ and $x \in X_b$, where the equation holds in X_c and $X_\gamma \hookrightarrow X_c$. Due to this equation, part (1.i), and $s(A^K) < 0$, the map

$$\lambda \mapsto \tilde{F}(\lambda) := (w-A)^{1-\beta-\gamma}R(\lambda, A) \in \mathcal{L}(X_b, X_c)$$

(initially defined for $\operatorname{Re}\lambda > \omega_0(A)$) can be extended to an analytic function on a halfplane $\mathbb{C}_{-\eta}$ for some $\eta > 0$ (possibly $\eta > 0$ has to be decreased). This means that

$$\lambda \mapsto F(\lambda) := (w-A)^c\tilde{F}(\lambda)(w-A)^{-b} = (w-A)^{1-\beta-b-\gamma+c}R(\lambda, A) \in \mathcal{L}(X) \qquad (5.3)$$

has an analytic extension to $\mathbb{C}_{-\eta}$. Set $-\delta = 1-\beta-b-\gamma+c \leq -1$. We temporarily use subrscripts to distinguish the different versions of A. By (5.3), the identity

$$(w-A)^{-\delta}x = (\lambda - A_{-1})F(\lambda)x, \qquad x \in X, \qquad (5.4)$$

holds in X_{-1} for $\operatorname{Re}\lambda > -\eta$. This can be rewritten as

$$(w - A_{-1})F(\lambda)x = (w - \lambda)F(\lambda)x + (w - A)^{-\delta}x.$$

Since the right hand side belongs to X, we obtain $F(\lambda)x \in D(A) = X_1$. Iterating this argument, one deduces $F(\lambda)X \subset X_{\delta+1}$. Formula (5.4) thus shows that $\lambda - A_\delta : X_{\delta+1} \to X_\delta$ is surjective for all $\lambda \in \mathbb{C}_{-\eta}$. If $(\lambda - A_\delta)x = 0$ for some $\lambda \in \mathbb{C}_{-\eta}$ and $x \in X_{\delta+1}$, then we obtain in the same way that $0 = F(\lambda)(\lambda - A_\delta)x = (w - A)^{-\delta}x$ which yields $x = 0$. As a result, $\mathbb{C}_{-\eta} \subseteq \rho(A_\delta) = \rho(A)$.
(2.i) Consequently, (5.1) is valid for $\lambda \in \mathbb{C}_{-\eta}$. We multiply (5.1) by $(1 + |\operatorname{Im}\lambda|^\iota)^{-1}$. Due to the assumptions and Lemma 3.3, the multiplied equation shows that the function

$$\lambda \mapsto (1 + |\operatorname{Im}\lambda|^\iota)^{-1}R(\lambda, A)(w - A)^{1-\beta-\gamma}B \;\in\; \mathcal{L}(U, X_c)$$

is bounded on $\mathbb{C}_{-\eta}$.
(2.ii) Also (5.2) is valid on $\mathbb{C}_{-\eta}$. This equation is multiplied by $(1 + |\operatorname{Im}\lambda|^\kappa)^{-1}(1 + |\operatorname{Im}\lambda|^\iota)^{-1}$. Then part (2.i), the assumptions, and Lemma 3.3, imply that the maps

$$\lambda \mapsto (1 + |\operatorname{Im}\lambda|^\kappa)^{-1}(1 + |\operatorname{Im}\lambda|^\iota)^{-1}(w - A)^{1-\beta-\gamma}R(\lambda, A) \;\in\; \mathcal{L}(X_b, X_c),$$
$$\lambda \mapsto (1 + |\operatorname{Im}\lambda|^\kappa)^{-1}(1 + |\operatorname{Im}\lambda|^\iota)^{-1}(w - A)^{-\delta}R(\lambda, A) \;\in\; \mathcal{L}(X)$$

are bounded on $\mathbb{C}_{-\eta}$. One can now conclude that $s_\alpha(A) < 0$ as in Lemma 3.2 of [13].

(b) If $\omega_{\beta+\gamma-1}(A) < 0$, it is clear that (A, B) is $(\beta+\gamma-1)$–stabilizable on every X_b (take $K = 0$) and that (A, C) is $(\beta + \gamma - 1)$–detectable on every X_c (take $J = 0$). Moreover, by Lemma 4.8 the operator mapping u to the function

$$z(t) = C(w - A)^{1-\gamma}R(w, A)\int_0^t T(t - s)(w - A)^{1-\beta}u(s)\, ds, \quad t \geq 0,$$

is bounded from $L^2(\mathbb{R}_+, U)$ to $L^2(\mathbb{R}_+, Y)$. Since this operator is clearly translation invariant, this fact implies the boundedness of $G(\lambda) = C(w - A)^{1-\beta-\gamma}R(\lambda, A)B$ for $\operatorname{Re}\lambda > -\varepsilon$ due to Theorem 3.1 and Remark 3.8 in [26]. □

Modifying part (a) of the above proof, one obtains the following facts.

Corollary 5.2. *Let X, Y, U be Banach spaces, $\beta, \gamma \geq 1$, $1 - \beta \leq b \leq 0$, $0 \leq c \leq \gamma - 1$, $\iota, \kappa \geq 0$, A be a generator on X, B be a β–admissible control operator for A, and C be a γ–admissible observation operator for A.*

(a) *Assume that (A, B) is κ–stabilizable on X_b and that the function $\lambda \mapsto R(\lambda, A)(w - A)^{1-\beta}B \in \mathcal{L}(U, X)$ has a bounded analytic continuation to $\mathbb{C}_{-\varepsilon}$ for some $\varepsilon > 0$. Then we have $s_{\beta+b+\kappa-1}(A) < 0$, and thus $\omega_{\beta+b+\kappa-2+2/p}(A) < 0$ if X has Fourier type $p \in [1, 2]$.*
(b) *Assume that (A, C) is ι–detectable on X_c, and that the function $\lambda \mapsto C(w - A)^{1-\gamma}R(\lambda, A) \in \mathcal{L}(X, Y)$ has a bounded analytic continuation to $\mathbb{C}_{-\varepsilon}$ for some $\varepsilon > 0$. Then we have $s_{\gamma-c+\iota-1}(A) < 0$, and thus $\omega_{\gamma-c+\iota-2+2/p}(A) < 0$ if X has Fourier type $p \in [1, 2]$.*

The spectral theory of semigroups (see Remark 2.1) allows improvement on the above results in certain cases:

Corollary 5.3. *Assume that the hypotheses of Theorem 5.1(a) or of Corollary 5.2 hold and that one of conditions in Remark 2.1 is satisfied. Then* $\omega_0(A) < 0$.

We stress that Theorem 5.1 shows that each orbit $T(\cdot)x$ starting in $x \in D((w - A)^{\beta+b+\gamma-c+\iota+\kappa-2+2/p})$ is exponentially stable. It would be optimal to have it for $x \in X$ (which is true in the setting of Corollary 5.3). The possible loss in regularity comes from four different sources. We may loose

(a) $\beta - 1 + b$ and $\gamma - 1 - c$ powers of $w - A$ if the degrees of generalized admissibility and stabilizability/detectability do not match;
(b) κ and ι powers of $w - A$ due to the weakened concepts of stabilizability and detectability, respectively;
(c) $\frac{2}{p} - 1$ powers of $w - A$ depending on the Fourier type $p \in [1, 2]$ of X;
(d) 1 power of $w - A$ since we use G instead of H (or $\tilde{H}$).

Observe that points (a) and (b) only occur if there is a certain additional irregularity in the problem (compared with systems being well–posed in the usual sense). In particular, we obtain $\alpha = 1$ if we have 0–stabilizability on $X_{1-\beta}$ and 0–detectability on $X_{\gamma-1}$, which is the most natural case. Item (c) does not occur if one can take a Hilbert space as the state space X. Point (d) is the price we pay for avoiding the (regularized) transfer function or the input–output operators.

Thus we can account for most of the difference between our results and the more standard theorems of the type

$$\textit{stabilizability, detectability, Hilbert space, } \sup_{\mathbb{C}_0} \|H(\lambda)\| < \infty \quad \Longrightarrow \quad \omega_0(A) < 0;$$

see, e.g., [7], [15], [17], [28], [20], [21]. It only remains to explain why we must require the boundedness of G on the halfplane $\mathbb{C}_{-\varepsilon}$ rather than just on $\mathbb{C}_0$. If G is bounded on $\mathbb{C}_0$, our proof establishes that $(w - A)^{-\alpha} R(\cdot, A)$ is bounded on $\mathbb{C}_0$. For $\alpha = 0$, a standard power series argument then shows that $s_0(A) < 0$ (and thus $\omega_0(A) < 0$ if X is a Hilbert space). This conclusion does not hold if $\alpha > 0$, as seen in the next example [12, Ex.3.1].

Example 5.4. Let $X = \ell^2$ and $A(x_n)_n = ((in - \frac{1}{n})x_n)_n$. Then $\sigma(A) = \{in - \frac{1}{n} : n \in \mathbb{N}\}$ and $R(\lambda, A)(x_n)_n = ((\lambda - in + \frac{1}{n})^{-1} x_n)_n$ for $\lambda \notin \sigma(A)$. Further, A generates a C_0–semigroup $T(\cdot)$ on X with $\|T(t)\| = 1$ for $t \geq 0$ and, hence, $\omega_0(A) = s(A) = s_1(A) = \omega_1(A) = 0$. On the other hand, for $\lambda \in \mathbb{C}_0$ we have

$$\|R(1, A)R(\lambda, A)\| = \sup_{n\in\mathbb{N}} |1 - in + \tfrac{1}{n}|^{-1} \, |\lambda - in + \tfrac{1}{n}|^{-1} \leq \sup_{n\in\mathbb{N}} \frac{n}{|1 - in + \frac{1}{n}|} \leq 1. \qquad \diamond$$

We use our main result to show that certain problems are not detectable. The following example could be generalized in various directions.

Example 5.5. Consider the weakly coupled wave equation with acceleration point sensing, cf. [3],

$$\begin{aligned}
&\partial_{tt}v(t,x) = \Delta v(t,x) - b\partial_t v(t,x) + \kappa w(t,x), \qquad t \geq 0,\ x \in \Omega, \\
&\partial_{tt}w(t,x) = \Delta w(t,x) + \kappa v(t,x), \qquad t \geq 0,\ x \in \Omega, \\
&v(t,x) = 0, \quad w(t,x) = 0, \qquad t \geq 0,\ x \in \partial\Omega, \\
&v(0,x) = v_0(x),\ \partial_t v(0,x) = v_1(x),\ w(0,x) = w_0(x),\ \partial_t w(0,x) = w_1(x), \quad x \in \Omega, \\
&y(t) = \partial_{tt}w(t,0) = \Delta w(t,0) + \kappa v(t,0), \qquad t \geq 0,
\end{aligned}$$

on the interval $\Omega = (-1,1)$. Here $b, \kappa > 0$ and κ is smaller than the absolute value of the first eigenvalue of the Dirichlet Laplacian Δ_D on $L^2(-1,1)$ with domain $H^2(-1,1) \cap H_0^1(-1,1)$. We set $X = H_0^1(-1,1) \times L^2(-1,1) \times H_0^1(-1,1) \times L^2(-1,1)$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \Delta_D & -b & \kappa & 0 \\ 0 & 0 & 0 & 1 \\ \kappa & 0 & \Delta_D & 0 \end{pmatrix}$$

with domain $D(A) = D(\Delta_D) \times H_0^1(-1,1) \times D(\Delta_D) \times H_0^1(-1,1)$. It is shown in [1] and [4, §4.2] that A generates a bounded C_0–semigroup $T(\cdot)$ on X, the spectrum of A belongs to the open left half plane, $s(A) = 0$, $R(\lambda, A)A^{-2}$ is bounded for $\operatorname{Re}\lambda \geq 0$, and $\|T(t)A^{-2}\| \leq c/t$. (This result does not require that the space dimension is equal to 1.) We further introduce the observation operator

$$C = (\kappa\delta_0, 0, \delta_0\Delta, 0) = (0,0,0,\delta_0)A$$

where $\delta_0 f = f(0)$. Since CA^{-2} is bounded, C is (at least) 3–admissible. Moreover, $CA^{-4}R(\lambda, A)$ is bounded for $\operatorname{Re}\lambda \geq 0$. Since $\omega_\alpha(A) = 0$ for all $\alpha \geq 0$, we deduce from Corollary 5.2 that the above problem is not 0–detectable on X_α for $\alpha \in [0,4]$. ◇

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