Functional Linear Models

Statistical Models

So far we have focussed on *exploratory data analysis*

- Smoothing
- Functional covariance
- Functional PCA

Now we wish to examine predictive relationships $\longrightarrow$ generalization of linear models.

Functional Linear Regression

$$y_i = \alpha + \mathbf{x}_i \beta + \epsilon_i$$

Three different scenarios for y_i $\mathbf{x}_i$

- Functional covariate, scalar response
- Scalar covariate, functional response
- Functional covariate, functional response

We will deal with each in turn.

Scalar Response Models

Generalization of multiple linear regression

$$y_i = \alpha + \sum \beta_j x_i(t_j) + \epsilon_i = \alpha + \mathbf{x}_i \beta + \epsilon_i$$

becomes

$$y_i = \alpha + \int \beta(t) x_i(t) dt + \epsilon_i$$

General trick: functional data model = multivariate model with sums replaced by integrals.

Identification

Problem:

- In linear regression, we must have fewer covariates than observations.
- If I have $y_i, x_i(t)$, there are *infinitely* many covariates.

$$y_i = \alpha + \int \beta(t)x_i(t)dt + \epsilon_i$$

Estimate β by minimizing squared error:

$$\beta(t) = \text{argmin} \sum \left(y_i - \alpha - \int \beta(t)x_i(t)dt \right)^2$$

But I can always make the $\epsilon_i = 0$.

Smoothing

Additional constraints: we want to insist that $\beta(t)$ is smooth.

Add a smoothing penalty:

$$\text{PENSSE}_{\lambda}(\beta) = \sum_{i=1}^{n} \left(y_i - \alpha - \int \beta(t) x_i(t) dt \right)^2 + \lambda \int [L\beta(t)]^2 \, dt$$

Very much like smoothing

Still need to represent $\beta(t)$ – use a basis expansion:

$$\beta(t) = \sum c_i \phi_i(t).$$

Calculation

$$y_i = \alpha + \int \beta(t) x_i(t) dt + \epsilon_i = \alpha + \left[\int \Phi(t) x_i(t) dt \right] \mathbf{c} + \epsilon_i$$
$$= \alpha + \mathbf{x}_i \mathbf{c} + \epsilon_i$$

for $\mathbf{x}_i = \int \Phi(t) x_i(t) dt$. With $Z_i = [1 \mathbf{x}_i]$,

$$\mathbf{y} = Z \begin{bmatrix} \alpha \\ \mathbf{c} \end{bmatrix} + \epsilon$$

and with smoothing penalty matrix R_L:

$$[\hat{\alpha} \ \hat{\mathbf{c}}^T]^T = \left(Z^T Z + \lambda R_L \right)^{-1} Z^T \mathbf{y}$$

Then

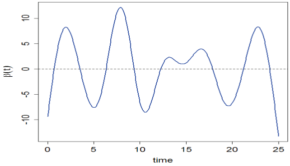
$$\hat{\mathbf{y}} = \int \hat{\beta}(t) x_i(t) dt = Z \begin{bmatrix} \hat{\alpha} \\ \hat{\mathbf{c}} \end{bmatrix} = S_\lambda \mathbf{y}$$

Choosing Smoothing Parameters

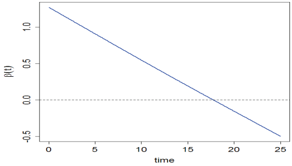
Cross-Validation:

$$\mathrm{CV}(\lambda) = \sum \left(\frac{y_i - \hat{y}_i}{1 - S_{ii}} \right)^2$$

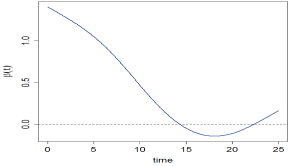
$\lambda = e^{-1}$



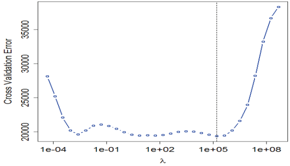
$\lambda = e^{20}$



$\lambda = e^{12}$



CV Error



Confidence Intervals

Assuming independent

$$\epsilon_i \sim N(0, \sigma_e^2)$$

We have that

$$\text{Var}\begin{bmatrix} \hat{\alpha} \\ \hat{\mathbf{c}} \end{bmatrix} = \left[\left(Z^T Z + \lambda R\right)^{-1} Z^T\right] \left[\sigma_e^2 \mathbb{I}\right] \left[Z\left(Z^T Z + \lambda R\right)^{-1}\right]$$

Estimate

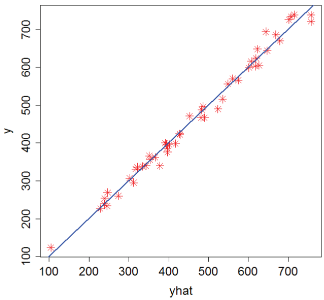
$$\hat{\sigma}_e^2 = SSE/(n - df), \;\; df = \text{trace}(S_\lambda)$$

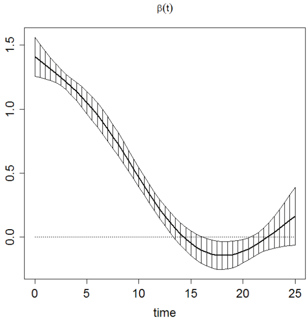
And (pointwise) confidence intervals for $\beta(t)$ are

$$\Phi(t)\hat{\mathbf{c}} \pm 2\sqrt{\Phi(t)^T \text{Var}[\hat{\mathbf{c}}]\Phi(t)}$$

Confidence Intervals

$R^2 = 0.987$ $\sigma^2 = 349$, df $= 5.04$





Extension to multiple functional covariates follows same lines:

$$y_i = \beta_0 + \sum_{j=1}^{p} \int \beta_j(t) x_{ij}(t) dt + \epsilon_i$$

functional Principal Components Regression

Alternative: principal components regression.

$$x_i(t) = \sum d_{ij}\xi_j(t) \quad d_{ij} = \int x_i(t)\xi_j(t)dt$$

Consider the model:

$$y_i = \beta_0 + \sum \beta_j d_{ij} + \epsilon_i$$

- Reduces to a standard linear regression problem.
- Avoids the need for cross-validation (assuming number of PCs is fixed).

fPCA and Functional Regression Interpretation

$$y_i = \beta_0 + \sum \beta_j d_{ij} + \epsilon_i$$

Recall that $d_{ij} = \int x_i(t)\xi_j(t)dt$ so

$$y_i = \beta_0 + \sum \int \beta_j \xi_j(t) x_i(t) dt + \epsilon_i$$

and we can interpret

$$\beta(t) = \sum \beta_j \xi_j(t)$$

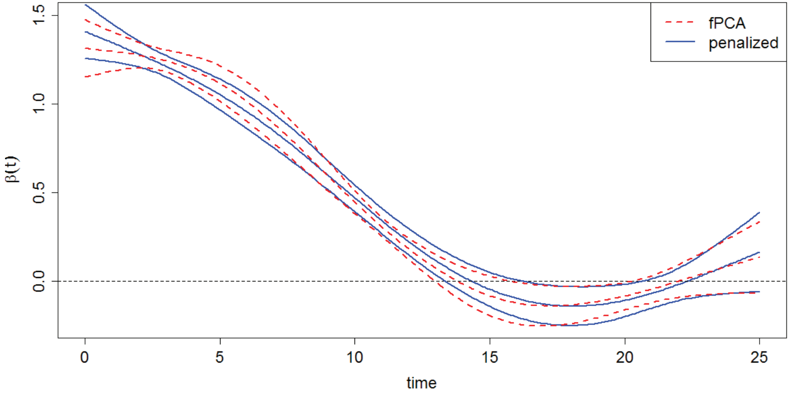
and write

$$y_i = \beta_0 + \int \beta(t) x_i(t) dt + \epsilon_i$$

Confidence intervals derive from variance of the d_{ij}.

A Comparison

Medfly Data: fPCA on 4 components ($R^2 = 0.988$) vs Penalized Smooth ($R^2 = 0.987$)



Advantages of FPCA-based approach

- Parsimonious description of functional data as it is the unique linear representation which explains the highest fraction of variance in the data with a given number of components.
- Main attraction is equivalence $X(\cdot) \sim (\xi_1, \xi_2, \cdots)$, so that $X(\cdot)$ can be expressed in terms of mean function and the sequence of eigenfunctions and uncorrelated FPC scores ξ_k's.
- For modeling functional regression: Functions $f\{X(\cdot)\}$ have an equivalent function $g(\xi_1, \xi_2, \cdots)$
- But need to pay prices
  - FPCs need to be estimated from data (finite sample)
  - Need to choose the number of FPCs

Two Fundamental Approaches

(Almost) all methods reduce to one of

1. Perform fPCA and use PC scores in a multivariate method.
2. Turn sums into integrals and add a smoothing penalty.

Applied in functional versions of

- generalized linear models
- generalized additive models
- survival analysis
- mixture regression
- ...

Both methods also apply to functional response models.

Functional Response Models

Functional Response Models

Case 1: Scalar Covariates: $(y_i(t), \mathbf{x}_i)$, most general linear model is

$$y_i(t) = \beta_0(t) + \sum_{j=1}^{p} \beta_i(t) x_{ij}.$$

Conduct a linear regression at each time t

But we might like to smooth; penalize *integrated squared error*

$$\text{PENSISE} = \sum_{i=1}^{n} \int (y_i(t) - \hat{y}_i(t))^2 \, dt + \sum_{j=0}^{p} \lambda_j \int [L_j \beta_j(t)]^2 \, dt$$

Usually keep λ_j, L_j all the same.

Concurrent Linear Model

Case 2: functional covariates

Extension of scalar covariate model: response only depends on $x(t)$ at the current time

$$y_i(t) = \beta_0(t) + \beta_1(t)x_i(t) + \epsilon_i(t)$$

- $y_i(t)$, $x_i(t)$ must be measured on same time domain.
- Must be appropriate to compare observations time-point by time-point
- Especially useful if $y_i(t)$ is a derivative of $x_i(t)$

Functional Response, Functional Covariate

General case: $y_i(t), x_i(s)$ - a functional linear regression at each time t:

$$y_i(t) = \beta_0(t) + \int \beta_1(s,t)x_i(s)ds + \epsilon_i(t)$$

- Same identification issues as scalar response models.
- Usually penalize β_1 in each direction separately

$$\lambda_s \int [L_s\beta_1(s,t)]^2 \, dsdt + \lambda_t \int [L_t\beta_1(s,t)]^2 \, dsdt$$

- Confidence Intervals etc. follow from same principles.

Summary

Three models

Scalar Response Models
- Functional covariate implies a functional parameter.
- Use smoothness of $\beta_1(t)$ to obtain identifiability.

Concurrent Linear Model
- $y_i(t)$ only depends on $x_i(t)$ at the current time.
- Scalar covariates = constant functions.

Functional Covariate/Functional Response
- Most general functional linear model.

Other Topics and Recent Developments

- Inference for functional regression models
- Dependent functional data
  - Multilevel/longitudinal/multivariate designs
- Registratoin
- Dynamics
- FDA for sparse longitudinal data
- More general/flexible regression models

Inference for functional regression models

Testing $H_0 : \beta(t) = 0$ under model

$$Y_i = \beta_0 + \int \beta(t) X_i(t)\, dt + \epsilon_i$$

- Penalized spline approach
  - $\beta(t) = \sum_{m=1}^{M} \eta_k B_k(t)$
- FPCA-based approach
  - data reduction: $(\xi_{i1}, \cdots, \xi_{iK})$
  - multivariate regression: $Y_i \sim \beta_1 \xi_{i1} + \cdots + \beta_K \xi_{iK}$
- Difficulty in inference arising from
  - regularization (smoothing)
  - choices of tuning parameters
  - accounting for uncertainly in two-step procedures

Penalized spline approach

- $H_0 : \eta_0 = \eta_1 = \cdots = \eta_M$
- Use roughness penalty $\lambda \int \beta(t)^2 \, dt$ to avoid overfitting
- Mixed model equivalence representation

$$Y_i = \beta_0 + \sum_{m=1}^{M} \eta_m V_{im} + \epsilon_i$$

$$(\eta_1, \cdots, \eta_M) \sim N(0, \sigma^2 \Sigma)$$

- Testing $H_0 : \sigma^2 = 0$
- Restricted LRT proposed in the literature.

Swihart, Goldsmith and Crainiceanu (2014). Restricted likelihood ratio tests for functional effects in the functional linear model. Technometrics, 56:483–493.

FPCA-based approach

- $Y_i \sim \beta_1 \xi_{i1} + \cdots + \beta_K \xi_{iK}$
- Testing $H_0 : \beta_1 = \cdots = \beta_K = 0$
- The number of PCs are chosen by
  - Percent of variance explained (PVE): e.g., 95% or 99%
  - Cross Validation
  - AIC, BIC
- Wald test

$$T = \sum_{k=1}^{K} \frac{\hat{\beta}_k^2}{\widehat{\text{var}}(\hat{\beta}_k)} = \frac{1}{n\hat{\sigma}_\epsilon^2} \sum_{k=1}^{K} \frac{Y^T \hat{\xi}_k \hat{\xi}_k^T Y}{\hat{\lambda}_k} \sim \chi_K^2$$

- *But is it a good idea to rank based on $X(t)$ only?*
  And how sensitive is the power to the choice of K?

FPCA-based approach

- Under alternative $\mathrm{H_a}: \beta_k = \beta_k$, where $\beta_k \neq 0$ for some k, it can be shown that $T \sim \chi^2_K(\eta)$, where

$$\eta = \frac{n}{\sigma^2_\epsilon} \sum_{k=1}^{K} \lambda_k \beta_k^2$$

- The power contribution of the k^{th} component depends on both λ_k and β_k
- We propose a new association-variation index (AVI): $AVI_k = \lambda_k \beta_k^2$
- Propose to rank and choose PCs based on AVI
- Asymptotics involves order statistics of χ^2_1 random variables

Su, Di and Hsu (2014). Hypothesis testing for functional linear models. Submitted.

FPCA-based approach

An example

Results with FA in RCST				
RCST	p-values		npc	
p(a)ve	AVI	V	AVI	V
0.50	0.0332	0.1007	2	2
0.85	0.0147	0.0637	3	5
0.99	0.0211	0.0035	5	10

- Standard FPCA approach sensitive to tuning parameter
- The new AVI-based approach is more robust

Dependent Functional Data

$$Y_{ij}(t) = X_{ij}(t) + \epsilon_{ij}(t)$$

- i: subject; j: visit
- $Y_{ij}(t)$ is recorded on $\Omega_{ij} = \{t_{ijs} : s = 1, 2, \cdots, T_{ij}\}$
- Functions from the same subject are correlated

$$Y_{ij}(t) = \mu(t) + Z_i(t) + W_{ij}(t) + \epsilon_{ij}(t)$$

- $Z_i(t)$'s and $W_{ij}(t)$'s are centered random functions
- Assume$Z_i(t)$ and $W_{ij}(t)$ are uncorrelated

Multilevel FPCA

KL expansion on both levels

$$Z_i(t) = \sum_{k=1}^{N_1} \xi_{ik}\, \phi_k^{(1)}(t)\,, \quad W_{ij}(t) = \sum_{l=1}^{N_2} \zeta_{ijl}\, \phi_l^{(2)}(t)$$

- $\phi_k^{(1)}(t), \phi_l^{(2)}(t)$: eigenfunctions
  dominating directions of variation at both levels
- ξ_{ik}, ζ_{ijl}: principal component scores
  magnitude of variation for each subject/visit
- $\lambda_k^{(1)} = \mathrm{var}(\xi_{ik}),\ \ \lambda_l^{(2)} = \mathrm{var}(\zeta_{ijl})$: eigenvalues
  the amount of variation explained

Multilevel FPCA

$$Y_{ij}(t) = \mu(t) + Z_i(t) + W_{ij}(t) + \epsilon_{ij}(t)$$

- Between subject level (level 1):
  $K_B(s,t) := \mathrm{cov}\{Z_i(s), Z_i(t)\} = \sum_{k=1}^{\infty} \lambda_k^{(1)} \phi_k^{(1)}(s)\, \phi_k^{(1)}(t)$
- Within subject level (level 2):
  $K_W(s,t) := \mathrm{cov}\{W_{ij}(s), W_{ij}(t)\} = \sum_{l=1}^{\infty} \lambda_l^{(2)} \phi_l^{(2)}(s)\, \phi_l^{(2)}(t)$
- Total: $K_T(s,t) := K_B(s,t) + K_W(s,t) + \sigma^2\, I(t=s)$

Note that

- $\mathrm{cov}\{Y_{ij}(s), Y_{ik}(t)\} = K_B(s,t) + \sigma^2\, I(t=s)$
- $\mathrm{cov}\{Y_{ij}(s), Y_{ij}(t)\} = K_B(s,t) + K_W(s,t) + \sigma^2\, I(t=s)$

MFPCA Algorithm

- Estimate $\mu(t)$ and $\eta_j(t)$ by univariate smoothing; estimate $K_T(s,t)$ and $K_B(s,t)$ via bivariate smoothing
- Set $\hat{K}_W(s,t) = \hat{K}_T(s,t) - \hat{K}_B(s,t)$
- Eigen-analysis of $\hat{K}_B(s,t)$ and $\hat{K}_W(s,t)$ to obtain $\hat{\lambda}_k^{(1)}$, $\hat{\phi}_k^{(1)}(t)$, $\hat{\lambda}_l^{(2)}$, $\hat{\phi}_l^{(2)}(t)$
- Estimate principal component scores

Note: we use penalized splines with REML for smoothing

R package "SemiPar"

Principal Component Scores

$$Y_{ij}(t) = \mu(t) + \sum_{k=1}^{N_1} \xi_{ik}\, \phi_k^{(1)}(t) + \sum_{l=1}^{N_2} \zeta_{ijl}\, \phi_l^{(2)}(t) + \epsilon_{ij}(t)$$

- Estimate scores, $\hat{\xi}_{ik}, \hat{\zeta}_{ijl}$, using *BLUP*
- Dimension reduction
  Subject level: $\{Y_{i1}(t), \cdots, Y_{iJ}(t)\} \to (\hat{\xi}_{i1}, \cdots, \hat{\xi}_{iN_1})$
- Predict individual curve $\hat{Y}_{ij}(t)$ with confidence bands
- Predict subject level curve $\hat{Z}_i(t)$ with confidence bands

Other extensions

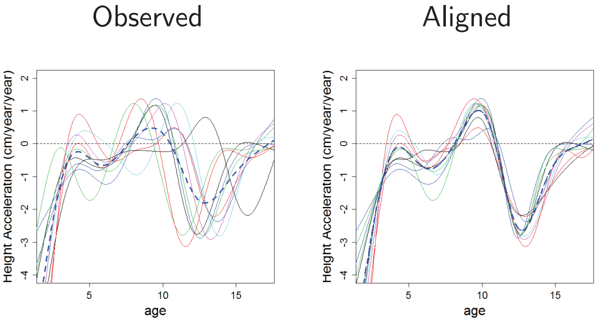
- Multilevel Functional Regression (Crainiceanu et al. 2009)
- Longitudinal/multivariate FPCA (more flexible correlations)

The Registration Problem

Most analyzes only account for variation in *amplitude*.

Frequently, observed data exhibit features that vary in *time*.

Berkeley Growth Acceleration



- Mean of unregistered curves has smaller peaks than any individual curve.
- Aligning the curves reduces variation by 25%

Defining a Warping Function

Requires a transformation of *time*.

Seek

$$s_i = w_i(t)$$

so that

$$\tilde{x}_i(t) = x_i(s_i)$$

are well aligned.

$w_i(t)$ are *time-warping* (also called *registration*) functions.

Landmark registration

For each curve $x_i(t)$ we choose points

$$t_{i1}, \dots, t_{iK}$$

We need a reference (usually one of the curves)

$$t_{01}, \dots, t_{0K}$$

so these define constraints

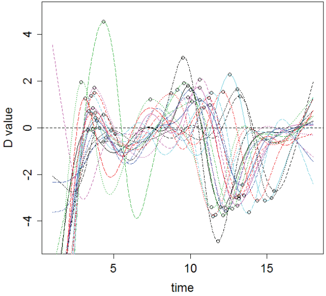
$$w_i(t_{ij}) = t_{0j}$$

Now we define a smooth function to go between these.

Identifying Landmarks

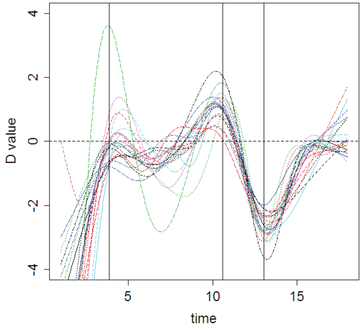
Major landmarks of interest:

- where $x_i(t)$ crosses some value
- location of peaks or valleys
- location of inflections

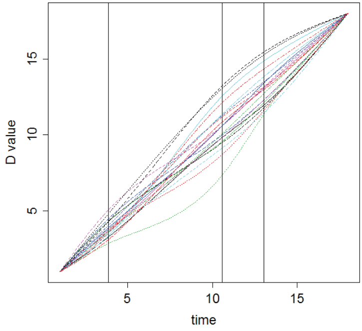


Results of Warping

Registered Acceleration



Warping Functions



Dynamics: Relationships between derivatives

Access to derivatives of functional data allows new models.

Variant on the concurrent linear model: e.g.

$$Dy_i(t) = \beta_0(t) + \beta_1(t)y_i(t) + \beta_2(t)x_i(t) + \epsilon_i(t)$$

Can be estimated like concurrent linear model.

But how do we understand these systems?

Principle Differential Analysis

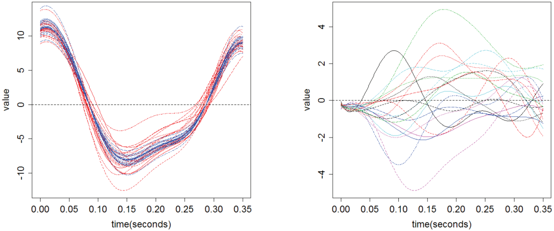
Translate autonomous dynamic model into linear differential operator:

$$Lx = D^2x + \beta_1(t)Dx(t) + \beta_0(t)x(t) = 0$$

Potential use in improving smooths (theory under development).

We can ask what is smooth? How does the data deviate from smoothness?

Solutions of $Lx(t) = 0$ Observed $Lx(t)$



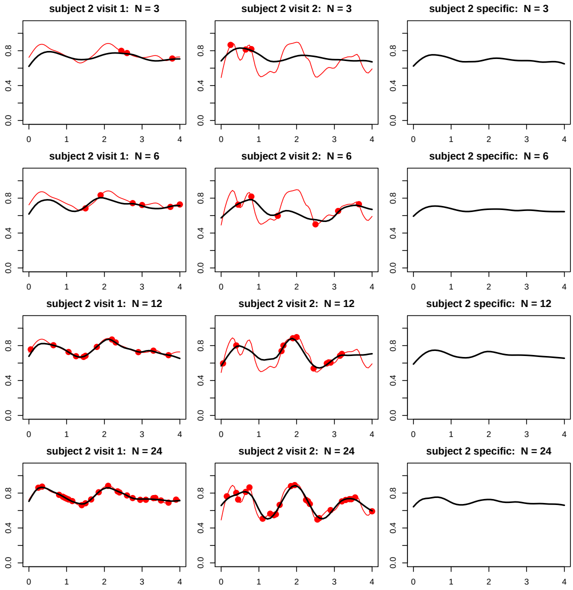
FDA for sparse longitudinal data

$$Y_ij = X_i(t_{ij}) + \epsilon_{ij}$$

- Data is recorded on sparse and irregular grid points $\Omega_i = \{t_{i1}, t_{i2}, \cdots, t_{in_i}\}$, n_i is small (bounded)
- But grid points are dense collectively, $\Omega = \cup_i \Omega_i$

- Difficulty of applying FDA techniques (e.g., FPCA)
  - Cannot pre-smooth trajectory for each subject
  - Estimation of FPC needs numerical integration $d_{ik} = \int \{x_i(t) - \mu(t)\}\phi_k(t)\, dt$
- Solution: Yao et al. (2005)
  - Pool all data, use (bivariate) smoothing
  - Estimate FPC by conditional expectations (BLUPs)

FDA for sparse longitudinal data



More general regression models

- Functional additive models (Muller et al., 2008; McLean et al., 2014)
- Partially functional linear regression (Kong et al., 2015)
- Functional mixture regression (Yao et al. 2011)
- ...

Recommended readings

- Yao, Muller and Wang(2005). Functional data analysis for sparse longitudinal data. JASA, 100: 577-590.
- Reiss and Ogden (2007). Functional Principal Component Regression and Functional Partial Least Squares. JASA, 102: 984–996.
- Ramsay, Hooker, Campbell, and Cao (2007). Parameter estimation for differential equations: a generalized smoothing approach. JRSS-B, 69: 741-796.
- Kneip and Ramsay (2008). Combining registration and fitting for functional models. JASA, 103(483), 1155-1165.
- Di, Crainiceanu, Caffo and Punjabi (2009). Multilevel Functional Principal Component Analysis. AOAS, 3: 458–488.
- Crainiceanu, Staicu and Di (2009). Generalized Multilevel Functional Regression. JASA, 104: 1550–1561.
- Senturk and Muller (2010). Functional varying coefficient models for longitudinal data. JASA, 105: 1256-1264.
- Goldsmith, Greven and Crainiceanu(2013). Corrected confidence bands for functional data using principal components. Biometrics, 69(1), 41-51.